ANALYTIC GEOMETRY

Study Guide

Georgia End-Of-Course Tests

August 26, 2013

Revision Log

08/26/2013:

Unit 7 (Applications of Probability)
page 200; Key Idea #7 – text and equation were updated to denote ***intersection*** rather than union
page 203; Review Example #2 Solution – union symbol has been corrected to ***intersection*** symbol in lines 3 and 4 of the Solution text

TABLE OF CONTENTS

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INTRODUCTION

This study guide is designed to help students prepare to take the Georgia End-of-Course Test (EOCT) for ***Analytic Geometry***. This study guide provides information about the EOCT, tips on how to prepare for it, and some suggested strategies students can use to perform their best.

What is the EOCT? The EOCT program was created to improve student achievement through effective instruction and assessment of the material in the state-mandated content standards. The EOCT program helps ensure that all Georgia students have access to rigorous courses that meet high academic expectations. The purpose of the EOCT is to provide diagnostic data that can be used to enhance the effectiveness of schools' instructional programs.

The Georgia End-of-Course Testing program is a result of the A+ Educational Reform Act of 2000, O.C.G.A. §20-2-281. This act requires the Georgia Department of Education to create end-of-course assessments for students in grades nine through twelve for the following core high school subjects:

Mathematics
- Mathematics II: Geometry/Algebra II/Statistics

--OR--
- GPS Geometry

--OR--
- Coordinate Algebra (beginning 2012–2013)
- Analytic Geometry (beginning 2013–2014)

Social Studies
- United States History
- Economics/Business/Free Enterprise

Science
- Biology
- Physical Science

English Language Arts
- Ninth Grade Literature and Composition
- American Literature and Composition

Getting started: The HOW TO USE THE STUDY GUIDE section on page 6 outlines the contents in each section, lists the materials you should have available as you study for the EOCT, and suggests some steps for preparing for the ***Analytic Geometry*** **EOCT**.

HOW TO USE THE STUDY GUIDE

This study guide is designed to help you prepare to take the ***Analytic Geometry* EOCT**. It will give you valuable information about the EOCT, explain how to prepare to take the EOCT, and provide some opportunities to practice for the EOCT. The study guide is organized into three sections. Each section focuses on a different aspect of the EOCT.

The OVERVIEW OF THE EOCT section on page 8 gives information about the test: dates, time, question format, and number of questions that will be on the ***Analytic Geometry* EOCT**. This information can help you better understand the testing situation and what you will be asked to do.

The PREPARING FOR THE EOCT section that begins on page 9 provides helpful information on study skills and general test-taking skills and strategies. It explains how to prepare before taking the test and what to do during the test to ensure the best test-taking situation possible.

The TEST CONTENT section that begins on page 15 explains what the ***Analytic Geometry* EOCT** specifically measures. When you know the test content and how you will be asked to demonstrate your knowledge, it will help you be better prepared for the EOCT. This section also contains some sample EOCT test questions, which are helpful for gaining an understanding of how a standard may be tested.

The activities in this guide are designed to be used by teachers and parents to help students with the ***Analytic Geometry* EOCT** but are not intended to be, nor should they be, used as a comprehensive guide to teaching and learning standards.

With some time, determination, and guided preparation, you will be better prepared to take the ***Analytic Geometry* EOCT**.

GET IT TOGETHER

In order to make the most of this study guide, you should have the following:

Materials:

* This study guide
* Pen or pencil
* Highlighter
* Paper

Resources:

* Classroom notes
* Mathematics textbook
* A teacher or other adult

Study Space:

* Comfortable (but not too comfortable)
* Good lighting
* Minimal distractions
* Enough work space

Time Commitment:

* When are you going to study?
* How long are you going to study?

Determination:

* Willingness to improve
* Plan for meeting

SUGGESTED STEPS FOR USING THIS STUDY GUIDE

1 Familiarize yourself with the structure and purpose of the study guide. (You should have already read the INTRODUCTION and HOW TO USE THE STUDY GUIDE. Take a few minutes to look through the rest of the study guide to become familiar with how it is arranged.)

2 Learn about the test and expectations of performance. (Read OVERVIEW OF THE EOCT.)

3 Improve your study skills and test-taking strategies. (Read PREPARING FOR THE EOCT.)

4 Learn what the test will assess by studying each unit and the strategies for answering questions that assess the standards in the unit. (Read TEST CONTENT.)

5 Answer the sample test question at the end of each lesson. Check your answer against the answer given to see how well you did. (See TEST CONTENT.)

OVERVIEW OF THE EOCT

Successful test takers understand the importance of knowing as much about a test as possible. This information can help you determine how to study and prepare for the EOCT and how to pace yourself during the test. The box below gives you a snapshot of the ***Analytic Geometry*** **EOCT** and other important information.

THE EOCT AT A GLANCE

Administration Dates:

The EOCT has three primary annual testing dates: once in the spring, once in the summer, and once in the winter. There are also mid-month online tests given in August, September, October, November, February, and March, as well as retest opportunities within the year.

Administration Time:

Each EOCT is composed of two sections, and students are given 60 minutes to complete each section. There is a short stretch break between the two sections of the test.

Question Format:

All the questions on the EOCT are multiple-choice.

Number of Questions:

Each section of the ***Analytic Geometry*** **EOCT** contains 31 questions; there are a total of 62 questions on the ***Analytic Geometry*** **EOCT**.

Impact on Course Grade:

For students in grade 10 or above beginning the 2011–2012 school year, the final grade in each course is calculated by weighing the course grade 85% and the EOCT score 15%. For students in grade 9 beginning the 2011–2012 school year, the final grade in each course is calculated by weighing the course grade 80% and the EOCT score 20%. A student must have a final grade of at least 70 to pass the course and to earn credit toward graduation.

If you have additional administrative questions regarding the EOCT, please visit the Georgia Department of Education Web site at www.doe.k12.ga.us, see your teacher, or see your school test coordinator.

PREPARING FOR THE EOCT

WARNING!

You cannot prepare for this kind of test in one night. Questions will ask you to apply your knowledge, not list specific facts. Preparing for the EOCT will take time, effort, and practice.

To do your best on the ***Analytic Geometry*** **EOCT**, it is important that you take the time necessary to prepare for this test and develop the skills that will help you take the EOCT.

First, you need to make the most of your classroom experiences and test-preparation time by using good **study skills**. Second, it is helpful to know general **test-taking strategies** to ensure that you will achieve your best score.

Study Skills

A LOOK AT YOUR STUDY SKILLS

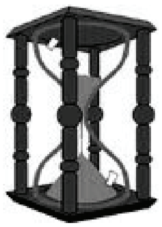
Before you begin preparing for this test, you might want to consider your answers to the following questions. You may write your answers here or on a separate piece of paper.

1. How would you describe yourself as a student?
   Response: ____
2. What are your study skills strengths and/or weaknesses as a student?
   Response: ____
3. How do you typically prepare for a mathematics test?
   Response: ____
4. Are there study methods you find particularly helpful? If so, what are they?
   Response: ____
5. Describe an ideal study situation (environment).
   Response: ____
6. Describe your actual study environment.
   Response: ____
7. What can you change about the way you study to make your study time more productive?
   Response: ____

Effective study skills for preparing for the EOCT can be divided into three categories:

- **Time Management**
- **Organization**
- **Active Participation**

Time Management



Do you have a plan for preparing for the EOCT? Often students have good intentions for studying and preparing for a test, but without a plan, many students fall short of their goals. Here are some strategies to consider when developing your study plan:

- Set realistic goals for what you want to accomplish during each study session and chart your progress.
- Study during your most productive time of the day.
- Study for reasonable amounts of time. Marathon studying is not productive.
- Take frequent breaks. Breaks can help you stay focused. Doing some quick exercises (e.g., sit-ups or jumping jacks) can help you stay alert.
- Be consistent. Establish your routine and stick to it.
- Study the most challenging test content first.
- For each study session, build in time to review what you learned in your last study session.
- Evaluate your accomplishments at the end of each study session.
- Reward yourself for a job well done.

Organization

You don’t want to waste your study time. Searching for materials, trying to find a place to study, and debating what and how to study can all keep you from having a productive study session. Get organized and be prepared. Here are a few organizational strategies to consider:

- Establish a study area that has minimal distractions.
- Gather your materials in advance.
- Develop and implement your study plan. (See Appendices A–D for sample study plan sheets.)

Active Participation

Students who actively study will learn and retain information longer. Active studying also helps you stay more alert and be more productive while learning new information. What is active studying? It can be anything that gets you to interact with the material you are studying. Here are a few suggestions:

- ♦ Carefully read the information and then DO something with it. Mark the important points with a highlighter, circle it with a pen, write notes on it, or summarize the information in your own words.
- ♦ Ask questions. As you study, questions often come into your mind. Write them down and actively seek the answers.
- ♦ Create sample test questions and answer them.
- ♦ Find a friend who is also planning to take the test and quiz each other.

Test-Taking Strategies

There are many test-taking strategies that you can use before and during a test to help you have the most successful testing situation possible. Below are a few questions to help you take a look at your test-taking skills.

A LOOK AT YOUR TEST-TAKING SKILLS

As you prepare to take the EOCT, you might want to consider your answers to the following questions. You may write your answers here or on your own paper.

1. How would you describe your test-taking skills?
   Response: ____

2. How do you feel when you are taking a test?
   Response: ____

3. List the strategies that you already know and use when you are taking a test.
   Response: ____

4. List test-taking behaviors you use that contribute to your success when preparing for and taking a test.
   Response: ____

5. What would you like to learn about taking tests?
   Response: ____

Suggested Strategies to Prepare for the EOCT

Learn from the past. Think about your daily/weekly grades in your mathematics classes (past and present) to answer the following questions:

- In which specific areas of mathematics were you or are you successful?

  Response: ____

- Is there anything that has kept you from achieving higher scores?

  Response: ____

- What changes should you implement to achieve higher scores?

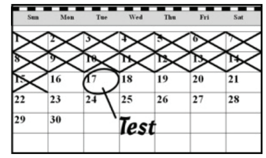
  Response: ____

Before taking the EOCT, work toward removing or minimizing any obstacles that might stand in the way of performing your best. The test preparation ideas and test-taking strategies in this section are designed to help guide you to accomplish this.

Be prepared. The best way to perform well on the EOCT is to be prepared. In order to do this, it is important that you know what standards/skills will be measured on the ***Analytic Geometry*** **EOCT** and then practice understanding and using those standards/skills. The TEST CONTENT section of this study guide is designed to help you understand the specific standards that are on the ***Analytic Geometry*** **EOCT** and give you suggestions for how to study the standards that will be assessed. Take the time to read through this material and follow the study suggestions. You can also ask your math teacher for any suggestions he or she might offer to prepare for the EOCT.

Start now. Don't wait until the last minute to start preparing. Begin early and pace yourself. By preparing a little bit each day, you will retain the information longer and increase your confidence level. Find out when the EOCT will be administered so that you can allocate your time appropriately.

Suggested Strategies the Day before the EOCT



✓ **Review what you learned from this study guide**.

1. Review the general test-taking strategies discussed in the "Top 10 Suggested Strategies during the EOCT" on page 14.
2. Review the content information discussed in the TEST CONTENT section beginning on page 15.
3. Focus your attention on the main topic, or topics, that you are most in need of improving.

✓ **Take care of yourself**.

1. Try to get a good night's sleep. Most people need an average of eight hours, but everyone's sleep needs are different.
2. Don't drastically alter your routine. If you go to bed too early, you might lie in bed thinking about the test. You want to get enough sleep so you can do your best.

Suggested Strategies the Morning of the EOCT

Eat a good breakfast. Choose foods high in protein for breakfast (and for lunch if the test is given in the afternoon). Some examples of foods high in protein are peanut butter, meat, and eggs. Protein gives you long-lasting, consistent energy that will stay with you through the test to help you concentrate better. Avoid foods high in sugar content. It is a misconception that sugar sustains energy—after an initial boost, sugar will quickly make you more tired and drained. Also, don't eat too much. A heavy meal can make you feel tired, so think about what you eat before the test.

Dress appropriately. If you are too hot or too cold during the test, it can affect your performance. It is a good idea to dress in layers, so you can stay comfortable, regardless of the room temperature, and keep your mind on the EOCT.

Arrive for the test on time. Racing late into the testing room can cause you to start the test feeling anxious. You want to be on time and prepared.

TOP 10

Suggested Strategies during the EOCT

These general test-taking strategies can help you do your best during the EOCT.

1 **Focus on the test.** Try to block out whatever is going on around you. Take your time and think about what you are asked to do. Listen carefully to all the directions.

2 **Budget your time.** Be sure that you allocate an appropriate amount of time to work on each question on the test.

3 **Take a quick break if you begin to feel tired.** To do this, put your pencil down, relax in your chair, and take a few deep breaths. Then, sit up straight, pick up your pencil, and begin to concentrate on the test again. Remember that each test section is only 60 minutes.

4 **Use positive self-talk.** If you find yourself saying negative things to yourself such as "I can't pass this test," it is important to recognize that you are doing this. Stop and think positive thoughts such as "I prepared for this test, and I am going to do my best." Letting the negative thoughts take over can affect how you take the test and your test score.

5 **Mark in your test booklet.** Mark key ideas or things you want to come back to in your test booklet. Remember that only the answers marked on your answer sheet will be scored.

6 **Read the entire question and the possible answer choices.** It is important to read the entire question so you know what it is asking. Read each possible answer choice. Do not mark the first one that "looks good."

7 **Use what you know.** Use what you have learned in class, from this study guide, and during your study sessions to help you answer the questions.

8 **Use content domain-specific strategies to answer the questions.** In the TEST CONTENT section, there are a number of specific strategies that you can use to help improve your test performance. Spend time learning these helpful strategies so that you can use them while taking the test.

9 **Think logically.** If you have tried your best to answer a question but you just aren't sure, use the process of elimination. Look at each possible answer choice. If it doesn't seem like a logical response, eliminate it. Do this until you've narrowed down your choices. If this doesn't work, take your best educated guess. It is better to mark something down than to leave it blank.

10 **Check your answers.** When you have finished the test, go back and check your work.

A WORD ON TEST ANXIETY

It is normal to have some stress when preparing for and taking a test. It is what helps motivate us to study and try our best. Some students, however, experience anxiety that goes beyond normal test "jitters." If you feel you are suffering from test anxiety that is keeping you from performing at your best, please speak to your school counselor, who can direct you to resources to help you address this problem.

TEST CONTENT

Up to this point in this study guide, you have been learning various strategies on how to prepare for and take the EOCT. This section focuses on what will be tested. It also includes sample questions that will let you apply what you have learned in your classes and from this study guide.

This section of the study guide will help you learn and review the various mathematical concepts that will appear on the ***Analytic Geometry*** **EOCT**. Since *mathematics* is a broad term that covers many different topics, the state of Georgia has divided it into five major **conceptual categories** that portray a coherent view of high school mathematics. Each of the conceptual categories is broken down into big ideas. These big ideas are called **content standards**. Each conceptual category contains standards that cover different ideas related to that category. Each question on the EOCT measures an individual standard within a conceptual category.

The five conceptual categories for the ***Analytic Geometry*** **EOCT** are the following:

- Geometry
- Algebra
- Functions
- Number and Quantity
- Statistics and Probability

These categories are important for several reasons. Together, they cover the major skills and concepts needed to understand and solve mathematical problems. These skills have many practical applications in the real world. Another more immediate reason they are important has to do with test preparation. The best way to prepare for any test is to study and know the material measured on the test.

This study guide is organized into seven **units** that review the material covered within the seven units of the Analytic Geometry curriculum map. It is presented by topic rather than by category or standard (although those are listed at the beginning of each unit and are integral to each topic). The more you understand about the topics in each unit, the greater your chances of getting a good score on the EOCT.

Studying the Content Standards and Topics
(Units 1 through 7)

You should be familiar with many of the content standards and topics that follow. It makes sense to spend more time studying the content standards and topics that you think may cause you problems. Even so, do not skip over any of them. The TEST CONTENT section has been organized into seven units. Each unit is organized by the following features:

- **Introduction:** an overview of what will be discussed in the unit.
- **Key Standards:** information about the specific standards that will be addressed. Standards that highlight mathematical modeling appear throughout the course and are highlighted with a (★) symbol. Strikethroughs in the standards are to highlight the portions that are not relevant to this course.
- **Main Topics:** the broad subjects covered in the unit.

  Each Main Topic includes:
  - **Key Ideas:** definitions of important words and ideas as well as descriptions, examples, and steps for solving problems.
  - **Review Examples:** problems with solutions showing possible ways to answer given questions.
  - **EOCT Practice Items:** sample multiple-choice questions similar to test items on the ***Analytic Geometry*** **EOCT** with answer keys provided.

With some time, determination, and guided preparation, you will be better prepared to take the ***Analytic Geometry*** **EOCT**.

Geometry Resource

(May include but are not limited to the following):

Properties of Congruence	
Reflexive Property	$\overline{AB} \cong \overline{AB}$; $\angle A \cong \angle A$
Symmetric Property	If $\overline{AB} \cong \overline{CD}$, then $\overline{CD} \cong \overline{AB}$. If $\angle A \cong \angle B$, then $\angle B \cong \angle A$.
Transitive Property	If $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{EF}$. If $\angle A \cong \angle B$ and $\angle B \cong \angle C$, then $\angle A \cong \angle C$.
Addition Postulates	
Segment Addition Postulate	If three points *A*, *B*, and *C* are collinear and *B* is between *A* and *C*, then $AB + BC = AC$.
Angle Addition Postulate	If point *B* is in the interior of $\angle AOC$, then $m\angle AOB + m\angle BOC = m\angle AOC$.
Angles	
Congruent Supplements Theorem	If two angles are supplementary to the same angle, then the two angles are congruent.
Congruent Complements Theorem	If two angles are complementary to the same angle, then the two angles are congruent.
Linear Pair Postulate	If two angles form a linear pair, then they are supplementary.
Right Angle Congruence Theorem	All right angles are congruent.
Vertical Angle Theorem	Vertical angles are congruent.
Parallel Lines	
Corresponding Angles Postulate	If two parallel lines are cut by a transversal, then the corresponding angles formed by the transversal are congruent.
Converse of Corresponding Angles Postulate	If two lines are cut by a transversal so that the corresponding angles formed by the transversal are congruent, then the lines are parallel.
Alternate Interior Angles Theorem	If two parallel lines are cut by a transversal, then the alternate interior angles formed by the transversal are congruent.
Converse of Alternate Interior Angles Theorem	If two lines are cut by a transversal so that the alternate interior angles formed by the transversal are congruent, then the lines are parallel.
Same-Side Interior Angles Theorem	If two parallel lines are cut by a transversal, then the same-side interior angles formed by the transversal are supplementary.
Converse of Same-Side Interior Angles Theorem	If two lines are cut by a transversal so that the same-side interior angles formed by the transversal are supplementary, then the lines are parallel.

Triangle Relationships	
Pythagorean Theorem	In a right triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse: $a^2 + b^2 = c^2$, where a and b are the lengths of the legs and c is the length of the hypotenuse of the triangle.
Converse of the Pythagorean Theorem	If $a^2 + b^2 = c^2$, then the triangle is a right triangle, where a and b are the lengths of the legs and c is the length of the hypotenuse of the triangle.
Pythagorean Inequality Theorems	If $c^2 > a^2 + b^2$, then the triangle is obtuse and if $c^2 < a^2 + b^2$, then the triangle is acute, where a and b are the lengths of the shorter sides and c is the length of the longest side of the triangle.
Side-Side-Side Congruence (SSS)	If three sides of a triangle are congruent to three sides of another triangle, then the triangles are congruent.
Side-Angle-Side Congruence (SAS)	If two sides and the included angle of a triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.
Angle-Side-Angle Congruence (ASA)	If two angles and the included side of a triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.
Angle-Angle-Side Congruence (AAS)	If two angles and a non-included side of a triangle are congruent to two angles and the corresponding non-included side of another triangle, then the triangles are congruent.
Hypotenuse-Leg Congruence (HL)	If the hypotenuse and a leg of one right triangle are congruent to the hypotenuse and a leg of another right triangle, then the triangles are congruent.
CPCTC	Corresponding parts of congruent triangles are congruent.
Angle-Angle Similarity (AA)	If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.
Side Splitter Theorem	If a line is parallel to a side of a triangle and intersects the other two sides, then it divides those two sides proportionally.
Converse of Side Splitter Theorem	If a line divides two sides of a triangle proportionally, then the line is parallel to the third side.
Third Angle Theorem	If two angles in one triangle are congruent to two angles in another triangle, then the third angles are also congruent.
Triangle Angle-Sum Theorem	The sum of the measures of the angles of a triangle is 180°.
Isosceles Triangle Theorem	If two sides of a triangle are congruent, then the angles opposite those sides are also congruent.
Converse of the Isosceles Triangle Theorem	If two angles of a triangle are congruent, then the sides opposite the angles are congruent.
Triangle Midsegment Theorem	If a segment joins the midpoints of two sides of a triangle, then the segment is parallel to the third side and half its length.

Triangle Inequality Theorem	The sum of the lengths of any two sides of a triangle is greater than the length of the third side.
Exterior Angle Theorem	The measure of each exterior angle of a triangle equals the sum of the measures of its two remote interior angles.
Parallelograms	
If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.	
If one pair of opposite sides of a quadrilateral are both congruent and parallel, then the quadrilateral is a parallelogram.	
If both pairs of opposite sides of a quadrilateral are congruent, then the quadrilateral is a parallelogram.	
If both pairs of opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.	
If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.	
If the diagonals of a parallelogram are congruent, then the parallelogram is a rectangle.	
Circles	
If a line is tangent to a circle, then it is perpendicular to the radius drawn to the point of tangency.	
Two segments tangent to a circle from a point outside the circle are congruent.	
A diameter that is perpendicular to a chord bisects the chord and its arc.	
In the same or congruent circles, congruent central angles have congruent arcs and congruent arcs have congruent central angles.	
In the same or congruent circles, congruent chords have congruent arcs and congruent arcs have congruent central angles.	
The measure of a central angle of a circle is equal to the measure of the intercepted arc.	
The measure of an inscribed angle is half the measure of its intercepted arc.	
An angle inscribed in a semicircle is a right angle.	
The measure of an angle formed by a tangent and a chord with its vertex on the circle is half the measure of the intercepted arc.	
When two chords intersect inside a circle, two pairs of vertical angles are formed. The measure of any one of the angles is half the sum of the measures of the arcs intercepted by the pair of vertical angles.	
When two chords intersect inside a circle, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord.	
Angles outside a circle can be formed by the intersection of two tangents, two secants, or a secant and a tangent. The formula for the measure of all three types of angles is half the difference of the measure of the larger intercepted arc and the measure of the smaller intercepted arc.	
When two secant segments intersect outside a circle, the product of the length of the segment outside the circle and the length of the entire secant segment is equal to the product of the length outside the circle and the length of the other secant segment.	
When a secant segment and a tangent segment intersect outside a circle, the product of the length of the segment of the secant outside the circle and the length of entire secant segment is equal to the square of the length of the tangent segment.	

Unit 1: Similarity, Congruence, and Proofs

This unit introduces the concepts of similarity and congruence. The definition of similarity is explored through dilation transformations. The concept of scale factor with respect to dilations allows figures to be enlarged or reduced. Rigid motions lead to the definition of congruence. Once congruence is established, various congruence criteria (e.g., ASA, SSS, and SAS) can be explored. Once similarity is established, various similarity criteria (e.g., AA) can be explored. These criteria, along with other postulates and definitions, provide a framework to be able to prove various geometric proofs. In this unit, various geometric figures are constructed. These topics allow students a deeper understanding of formal reasoning, which will be beneficial throughout the remainder of Analytic Geometry.

KEY STANDARDS

Understand similarity in terms of similarity transformations

MCC9-12.G.SRT.1 Verify experimentally the properties of dilations given by a center and a scale factor:

a. A dilation takes a line not passing through the center of the dilation to a parallel line, and leaves a line passing through the center unchanged.
b. The dilation of a line segment is longer or shorter in the ratio given by the scale factor.

MCC9-12.G.SRT.2 Given two figures, use the definition of similarity in terms of similarity transformations to decide if they are similar; explain using similarity transformations the meaning of similarity for triangles as the equality of all corresponding pairs of angles and the proportionality of all corresponding pairs of sides.

MCC9-12.G.SRT.3 Use the properties of similarity transformations to establish the AA criterion for two triangles to be similar.

Prove theorems involving similarity

MCC9-12.G.SRT.4 Prove theorems about triangles. Theorems include: a line parallel to one side of a triangle divides the other two proportionally, and conversely; the Pythagorean Theorem proved using triangle similarity.

MCC9-12.G.SRT.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

Understand congruence in terms of rigid motions

MCC9-12.G.CO.6 Use geometric descriptions of rigid motions to transform figures and to predict the effect of a given rigid motion on a given figure; given two figures, use the definition of congruence in terms of rigid motions to decide if they are congruent.

MCC9-12.G.CO.7 Use the definition of congruence in terms of rigid motions to show that two triangles are congruent if and only if corresponding pairs of sides and corresponding pairs of angles are congruent.

MCC9-12.G.CO.8 Explain how the criteria for triangle congruence (ASA, SAS, and SSS) follow from the definition of congruence in terms of rigid motions.

Prove geometric theorems

MCC9-12.G.CO.9 Prove theorems about lines and angles. Theorems include: vertical angles are congruent; when a transversal crosses parallel lines, alternate interior angles are congruent and corresponding angles are congruent; points on a perpendicular bisector of a line segment are exactly those equidistant from the segment's endpoints.

MCC9-12.G.CO.10 Prove theorems about triangles. Theorems include: measures of interior angles of a triangle sum to 180 degrees; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.

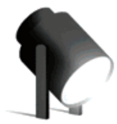
MCC9-12.G.CO.11 Prove theorems about parallelograms. Theorems include: opposite sides are congruent, opposite angles are congruent, the diagonals of a parallelogram bisect each other, and conversely, rectangles are parallelograms with congruent diagonals.

Make geometric constructions

MCC9-12.G.CO.12 Make formal geometric constructions with a variety of tools and methods (compass and straightedge, string, reflective devices, paper folding, dynamic geometric software, etc.). Copying a segment; copying an angle; bisecting a segment; bisecting an angle; constructing perpendicular lines, including the perpendicular bisector of a line segment; and constructing a line parallel to a given line through a point not on the line.

MCC9-12.G.CO.13 Construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle.

UNDERSTAND SIMILARITY IN TERMS OF SIMILARITY TRANSFORMATIONS

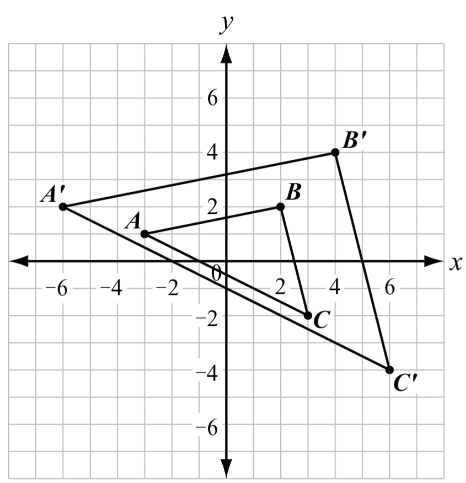


KEY IDEAS

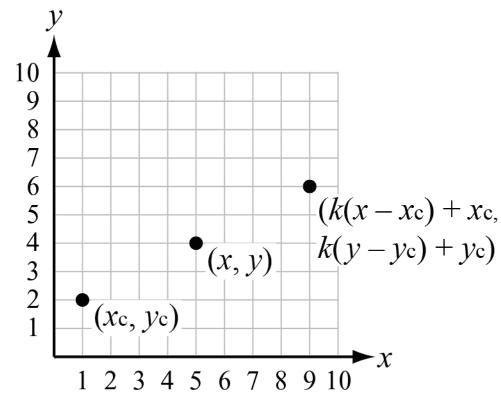
1. A ***dilation*** is a transformation that makes a figure larger or smaller than the original figure based on a ratio given by a ***scale factor***. When the scale factor is greater than 1, the figure is made larger. When the scale factor is between 0 and 1, the figure is made smaller. When the scale factor is 1, the figure does not change. When the center of dilation is the origin, you can multiply each coordinate of the original figure, or ***pre-image***, by the scale factor to find the coordinates of the dilated figure, or ***image***.

Example:

The diagram below shows $\triangle ABC$ dilated about the origin with a scale factor of 2 to create $\triangle A'B'C'$.



2. When the center of dilation is not the origin, you can use a rule that is derived from shifting the center of dilation, multiplying the shifted coordinates by the scale factor, and then shifting the center of dilation back to its original location. For a point (x, y) and a center of dilation (x_c, y_c), the rule for finding the coordinates of the dilated point with a scale factor of k is $(k(x - x_c) + x_c, k(y - y_c) + y_c)$.



When a figure is transformed under a dilation, the ***corresponding angles*** of the pre-image and the image have equal measures.

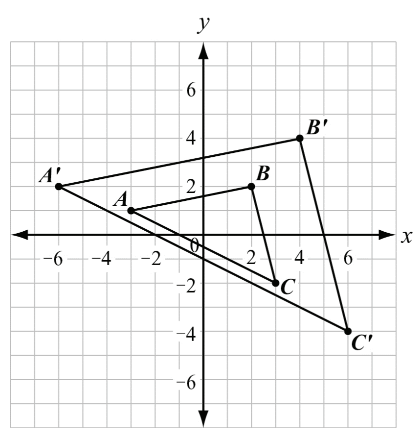
For $\triangle ABC$ and $\triangle A'B'C'$ below, $\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$.

When a figure is transformed under a dilation, the ***corresponding sides*** of the pre-image and the image are proportional.

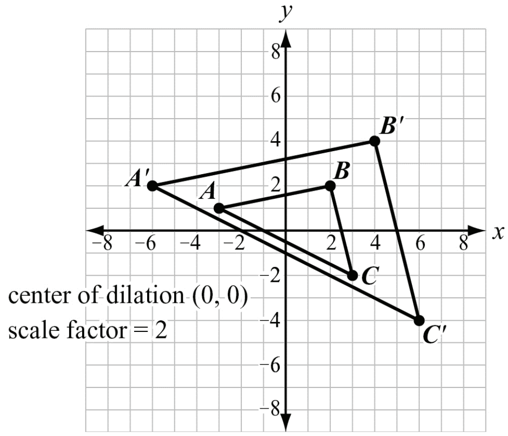
For $\triangle ABC$ and $\triangle A'B'C'$ below, $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.

So, when a figure is under a dilation transformation, the pre-image and the image are *similar*.

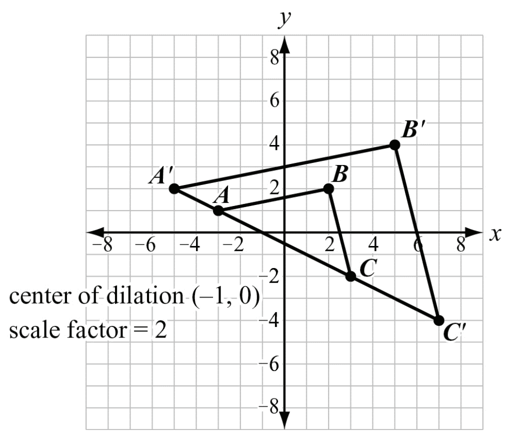
For $\triangle ABC$ and $\triangle A'B'C'$ below, $\triangle ABC \sim \triangle A'B'C'$.



3. When a figure is dilated, a segment of the pre-image that does not pass through the center of dilation is parallel to its image. In the figure below, $\overline{AC} \parallel \overline{A'C'}$ since neither segment passes through the center of dilation. The same is true about $\overline{AB}$ and $\overline{A'B'}$ as well as $\overline{BC}$ and $\overline{B'C'}$.



When the segment of a figure does pass through the center of dilation, the segment of the pre-image and image are on the same line. In the figure below, the center of dilation is on $\overline{AC}$, so $\overline{AC}$ and $\overline{A'C'}$ are on the same line.

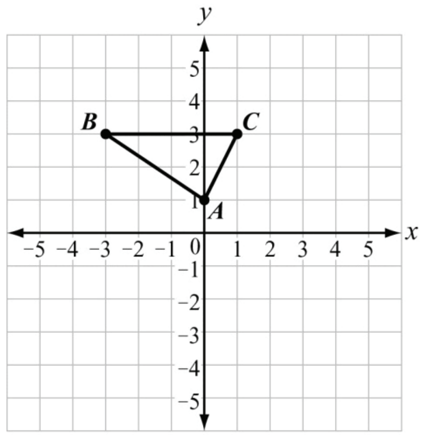


REVIEW EXAMPLES

1) Draw a triangle with vertices at $A(0, 1)$, $B(–3, 3)$, and $C(1, 3)$. Dilate the triangle using a scale factor of 1.5 and a center of (0, 0). Name the dilated triangle $A'B'C'$.

Solution:

Plot points $A(0, 1)$, $B(–3, 3)$, and $C(1, 3)$. Draw $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.



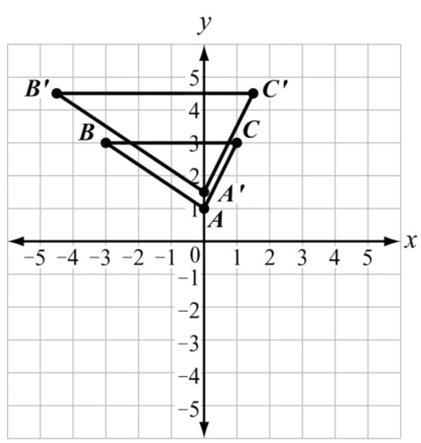
The center of dilation is the origin, so to find the coordinates of the image, multiply the coordinates of the pre-image by the scale factor 1.5.

Point $A': (1.5 \bullet 0, 1.5 \bullet 1) = (0, 1.5)$

Point $B': (1.5 \bullet (-3), 1.5 \bullet 3) = (-4.5, 4.5)$

Point $C': (1.5 \bullet 1, 1.5 \bullet 3) = (1.5, 4.5)$

Plot points $A'(0, 1.5)$, $B'(–4.5, 4.5)$, and $C'(1.5, 4.5)$. Draw $\overline{A'B'}$, $\overline{A'C'}$, and $\overline{B'C'}$.



Note: Since no part of the pre-image passes through the center of dilation, $\overline{BC} \parallel \overline{B'C'}$, $\overline{AB} \parallel \overline{A'B'}$, and $\overline{AC} \parallel \overline{A'C'}$.

2) Line segment *CD* is 5 inches long. If line segment *CD* is dilated to form line segment $C'D'$ with a scale factor of 0.6, what is the length of line segment $C'D'$?

Solution:

The ratio of the length of the image and the pre-image is equal to the scale factor.

$$\frac{C'D'}{CD} = 0.6$$

Substitute 5 for *CD*.

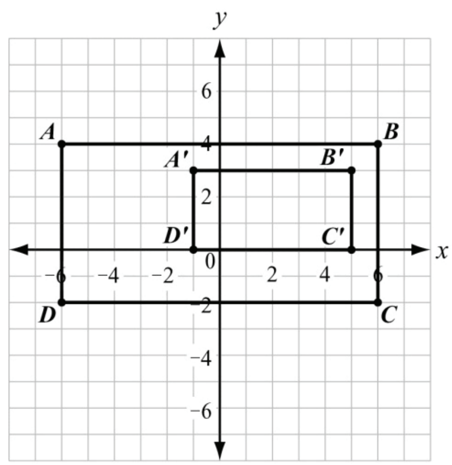
$$\frac{C'D'}{5} = 0.6$$

Solve for $C'D'$.

$$C'D' = 0.6 \bullet 5$$
$$C'D' = 3$$

The length of line segment $C'D'$ is 3 inches.

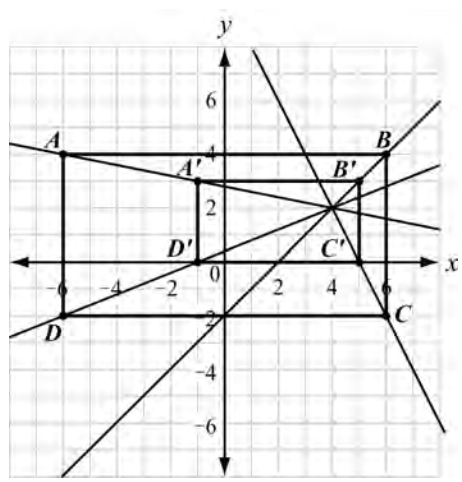
3) Figure $A'B'C'D'$ is a dilation of figure *ABCD*.



a. Determine the center of dilation.
b. Determine the scale factor of the dilation.
c. What is the relationship between the sides of the pre-image and corresponding sides of the image?

Solution:

a. To find the center of dilation, draw lines connecting each corresponding vertex from the pre-image to the image. The lines meet at the center of dilation.



The center of dilation is (4, 2).

b. Find the ratios of the lengths of the corresponding sides.

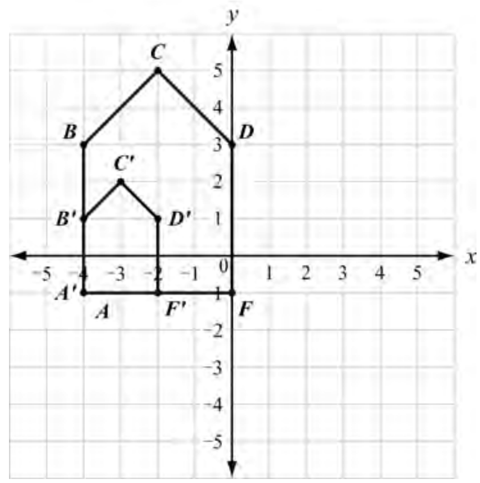
$$\frac{A'B'}{AB} = \frac{6}{12} = \frac{1}{2}$$
$$\frac{B'C'}{BC} = \frac{3}{6} = \frac{1}{2}$$
$$\frac{C'D'}{CD} = \frac{6}{12} = \frac{1}{2}$$
$$\frac{A'D'}{AD} = \frac{3}{6} = \frac{1}{2}$$

The ratio for each pair of corresponding sides is $\frac{1}{2}$, so the scale factor is $\frac{1}{2}$.

c. Each side of the image is parallel to the corresponding side of its pre-image and is $\frac{1}{2}$ the length.

EOCT Practice Items

1) Figure $A'B'C'D'F'$ is a dilation of figure $ABCDF$ by a scale factor of $\frac{1}{2}$. The dilation is centered at (–4, –1).



Which statement is true?

A. $\frac{AB}{A'B'} = \frac{B'C'}{BC}$

B. $\frac{AB}{A'B'} = \frac{BC}{B'C'}$

C. $\frac{AB}{A'B'} = \frac{BC}{D'F'}$

D. $\frac{AB}{A'B'} = \frac{D'F'}{BC}$

[Key: B]

2) Which transformation results in a figure that is similar to the original figure but has a greater area?

A. a dilation of $\triangle QRS$ by a scale factor of 0.25

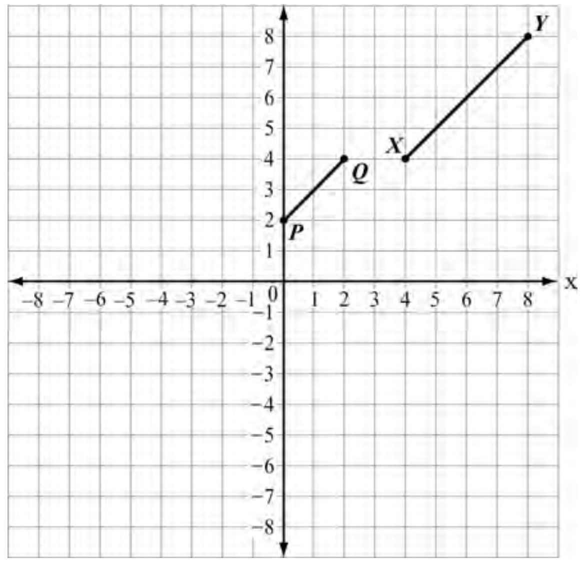
B. a dilation of $\triangle QRS$ by a scale factor of 0.5

C. a dilation of $\triangle QRS$ by a scale factor of 1

D. a dilation of $\triangle QRS$ by a scale factor of 2

[Key: D]

3) In the coordinate plane, segment $\overline{PQ}$ is the result of a dilation of segment $\overline{XY}$ by a scale factor of $\frac{1}{2}$.



Which point is the center of dilation?

A. (–4, 0)
B. (0, –4)
C. (0, 4)
D. (4, 0)

[Key: A]

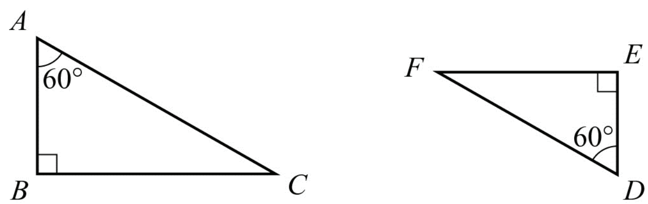
PROVE THEOREMS INVOLVING SIMILARITY

KEY IDEAS

1. When proving that two triangles are similar, it is sufficient to show that two pairs of corresponding angles of the triangles are congruent. This is called ***Angle-Angle* (*AA*) *Similarity***.

   Example:

   The triangles below are similar by AA Similarity because each triangle has a 60° angle and a 90° angle. The similarity statement is written as $\triangle ABC \sim \triangle DEF$, and the order in which the vertices are written indicates which angles/sides correspond to each other.



2. When a triangle is dilated, the pre-image and the image are similar triangles. There are three cases of triangles being dilated:
   - The image is congruent to the pre-image (scale factor of 1).
   - The image is smaller than the pre-image (scale factor between 0 and 1).
   - The image is larger than the pre-image (scale factor greater than 1).

3. When two triangles are **similar**, all corresponding pairs of angles are congruent.

4. When two triangles are **similar**, all corresponding pairs of sides are proportional.

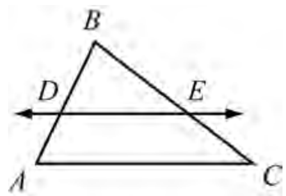
5. When two triangles are **congruent**, the triangles are also similar.

6. A ***two-column proof*** is a series of statements and reasons often displayed in a chart that works from given information to the statement that needs to be proven. Reasons can be given information, based on definitions, or based on postulates or theorems.

7. A ***paragraph proof*** also uses a series of statements and reasons that works from given information to the statement that needs to be proven, but the information is presented as running text in paragraph form.

REVIEW EXAMPLES

1) In the triangle shown, $\overline{AC} \parallel \overleftrightarrow{DE}$.

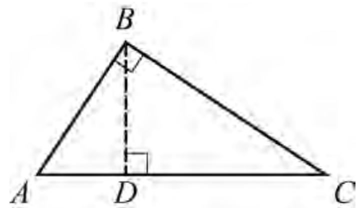


Prove that $\overleftrightarrow{DE}$ divides $\overline{AB}$ and $\overline{CB}$ proportionally.

Solution:

Step	Statement	Justification
1	$\overleftrightarrow{DE} \parallel \overline{AC}$	Given
2	$\angle BDE \cong \angle BAC$	If two parallel lines are cut by a transversal, then corresponding angles are congruent.
3	$\angle DBE \cong \angle ABC$	***Reflexive Property of Congruence*** because they are the same angle.
4	$\triangle DBE \sim \triangle ABC$	Angle-Angle (AA) Similarity
5	$\frac{BA}{BD} = \frac{BC}{BE}$	Corresponding sides of similar triangles are proportional.
6	$BD + DA = BA$ $BE + EC = BC$	Segment Addition Postulate
7	$\frac{BD + DA}{BD} = \frac{BE + EC}{BE}$	Substitution
8	$\frac{BD}{BD} + \frac{DA}{BD} = \frac{BE}{BE} + \frac{EC}{BE}$	Rewrite each fraction as sum of two fractions.
9	$1 + \frac{DA}{BD} = 1 + \frac{EC}{BE}$	Substitution
10	$\frac{DA}{BD} = \frac{EC}{BE}$	Subtraction Property of Equality
11	$\overleftrightarrow{DE}$ divides $\overline{AB}$ and $\overline{CB}$ proportionally	Definition of proportionality

2) Gale is trying to prove the Pythagorean Theorem using similar triangles. Part of her proof is shown below.



Step	Statement	Justification
1	$\angle ABC \cong \angle BDC$	All right angles are congruent.
2	$\angle ACB \cong \angle BCD$	Reflexive Property of Congruence
3	$\triangle ABC \sim \triangle BDC$	Angle-Angle (AA) Similarity
4	$\frac{BC}{DC} = \frac{AC}{BC}$	Corresponding sides of similar triangles are proportional.
5	$BC^2 = AC \bullet DC$	In a proportion, the product of the means equals the product of the extremes.
6	$\angle ABC \cong \angle ADB$	All right angles are congruent.
7	$\angle BAC \cong \angle DAB$	Reflexive Property of Congruence
8	$\triangle ABC \sim \triangle ADB$	Angle-Angle (AA) Similarity
9	$\frac{AB}{AD} = \frac{AC}{AB}$	Corresponding sides of similar triangles are proportional.
10	$AB^2 = AC \bullet AD$	In a proportion, the product of the means equals the product of the extremes.

What should Gale do to finish her proof?

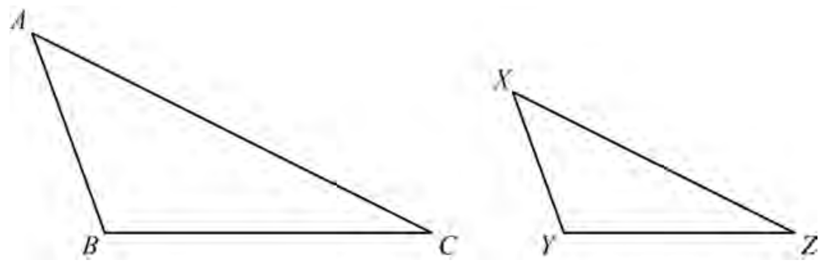
Solution:

Step	Statement	Justification
11	$AB^2 + BC^2 = AC \bullet AD + AC \bullet DC$	Addition
12	$AB^2 + BC^2 = AC(AD + DC)$	Distributive property
13	$AC = AD + DC$	Segment Addition Property
14	$AB^2 + BC^2 = AC \bullet AC$	Substitution
15	$AB^2 + BC^2 = AC^2$	Definition of exponent

$AB^2 + BC^2 = AC^2$ is a statement of the Pythagorean Theorem, so Gale's proof is complete.

EOCT Practice Items

1) In the triangles shown, $\triangle ABC$ is dilated by a factor of $\frac{2}{3}$ to form $\triangle XYZ$.

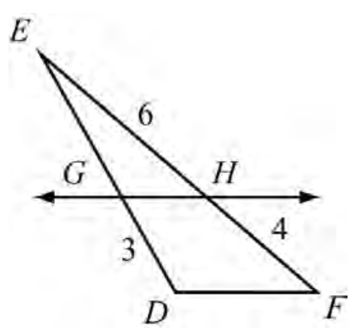


Given that $m\angle A = 50°$ and $m\angle B = 100°$, what is $m\angle Z$?

A. 15°
B. 25°
C. 30°
D. 50°

[Key: C]

2) In the triangle shown, $\overleftrightarrow{GH} \parallel \overline{DF}$.

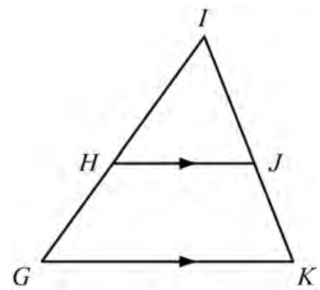


What is the length of $\overline{GE}$?

A. 2.0
B. 4.5
C. 7.5
D. 8.0

[Key: B]

3) Use this triangle to answer the question.



This is a proof of the statement "If a line is parallel to one side of a triangle and intersects the other two sides at distinct points, then it separates these sides into segments of proportional lengths."

	Step	Justification
1	$\overline{GK}$ is parallel to $\overline{HJ}$	Given
2	$\angle HGK \cong \angle IHJ$ $\angle IKG \cong \angle IJH$	?
3	$\triangle GIK \sim \triangle HIJ$	AA similarity postulate
4	$\frac{IG}{IH} = \frac{IK}{IJ}$	Corresponding sides of similar triangles are proportional
5	$\frac{HG + IH}{IH} = \frac{JK + IJ}{IJ}$	Segment addition postulate
6	$\frac{HG}{IH} = \frac{JK}{IJ}$	Subtraction property

Which reason justifies Step 2?

A. Alternate interior angles are congruent.
B. Alternate exterior angles are congruent.
C. Corresponding angles are congruent.
D. Vertical angles are congruent.

[Key: C]

UNDERSTAND CONGRUENCE IN TERMS OF RIGID MOTIONS

KEY IDEAS

1. A ***rigid motion*** is a transformation of points in space consisting of a sequence of one or more translations, reflections, and/or rotations (in any order). This transformation leaves the size and shape of the original figure unchanged.

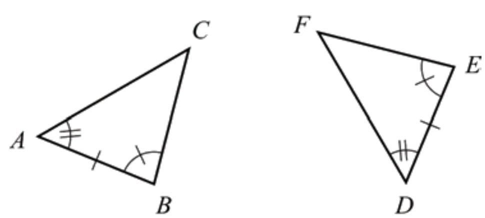
2. Two plane or solid figures are congruent if one can be obtained from the other by rigid motion (a sequence of rotations, reflections, and translations). ***Congruent figures*** have the same side lengths and same angle measures as each other.

3. Two triangles are congruent if and only if their corresponding sides and corresponding angles are congruent. This is sometimes referred to as ***CPCTC***, which means **C**orresponding **P**arts of **C**ongruent **T**riangles are **C**ongruent.

4. When given two congruent triangles, you can use a series of transformations, reflections, and rotations to show the triangles are congruent.

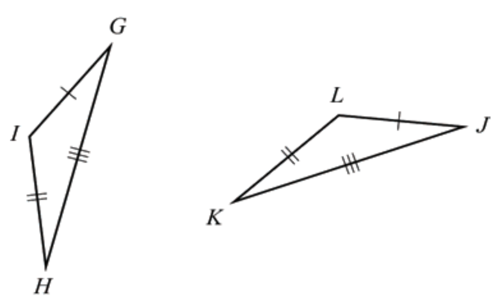
5. You can use ***ASA (Angle-Side-Angle)*** to show two triangles are congruent. If two angles and the included side of a triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

   $\triangle ABC \cong \triangle DEF$ by ASA.



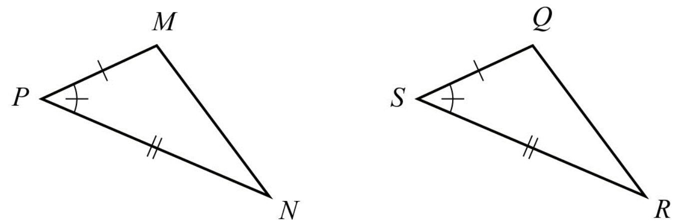
6. You can use ***SSS (Side-Side-Side)*** to show two triangles are congruent. If three sides of a triangle are congruent to three sides of another triangle, then the triangles are congruent.

   $\triangle GIH \cong \triangle JLK$ by SSS.



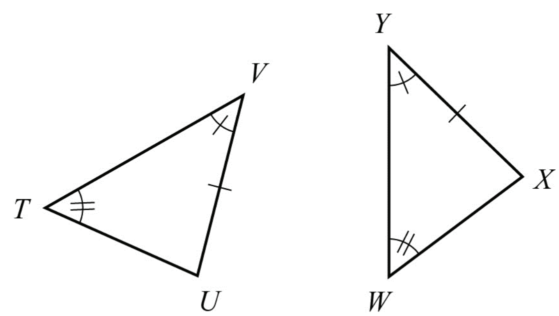
7. You can use ***SAS (Side-Angle-Side)*** to show two triangles are congruent. If two sides and the included angle of a triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

$\triangle MPN \cong \triangle QSR$ by SAS.



8. You can use ***AAS (Angle-Angle-Side)*** to show two triangles are congruent. If two angles and a non-included side of a triangle are congruent to two angles and the corresponding non-included side of another triangle, then the triangles are congruent.

$\triangle VTU \cong \triangle YWX$ by AAS.

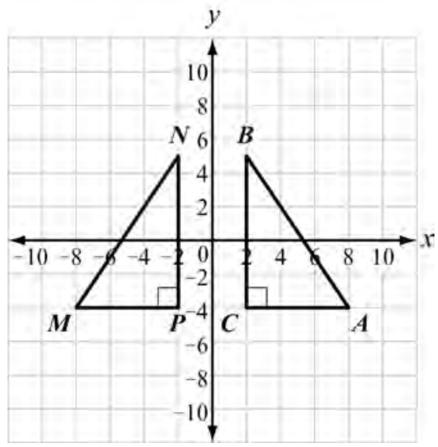


Important Tips

- If two sides and a ***non-included angle*** of one triangle are congruent to two sides and a non-included angle of a second triangle, the triangles are not necessarily congruent. Therefore, there is no way to show triangle congruency by Side-Side-Angle (SSA).
- If two triangles have all three angles congruent to each other, the triangles are similar, but not necessarily congruent. Thus, you can show similarity by Angle-Angle-Angle (AAA), but you cannot show congruence by AAA.

REVIEW EXAMPLES

1) Is $\triangle ABC$ congruent to $\triangle MNP$? Explain.



(scale unit = 2)

Solution:

$\overline{AC}$ corresponds to $\overline{MP}$. Both segments are 6 units long. $\overline{BC}$ corresponds to $\overline{NP}$. Both segments are 9 units long. Angle C (the included angle of $\overline{AC}$ and $\overline{BC}$) corresponds to angle P (the included angle of $\overline{MP}$ and $\overline{NP}$). Both angles measure 90°. Because two sides and an included angle are congruent, the triangles are congruent by SAS.

Or, $\triangle ABC$ is a reflection of $\triangle MNP$ over the *y*-axis. This means that all of the corresponding sides and corresponding angles are congruent, so the triangles are congruent. (Reflections preserve angle measurement and lengths, therefore, corresponding angles and sides are congruent.)

2) Rectangle *WXYZ* has coordinates *W*(1, 2), *X*(3, 2), *Y*(3, –3), and *Z*(1, –3).

a. Graph the image of rectangle *WXYZ* after a rotation of 90° clockwise about the origin. Label the image $W'X'Y'Z'$.

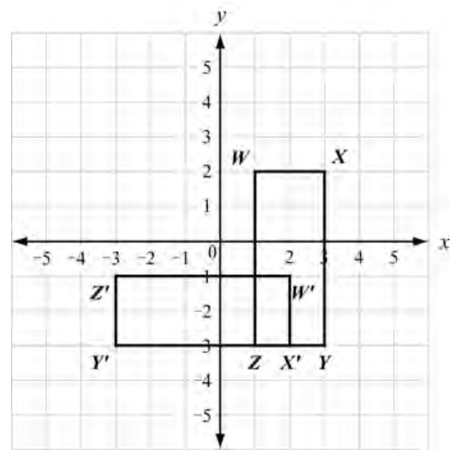
b. Translate rectangle $W'X'Y'Z'$ 2 units left and 3 units up.

c. Is rectangle *WXYZ* congruent to rectangle $W''X''Y''Z''$? Explain.

Solution:

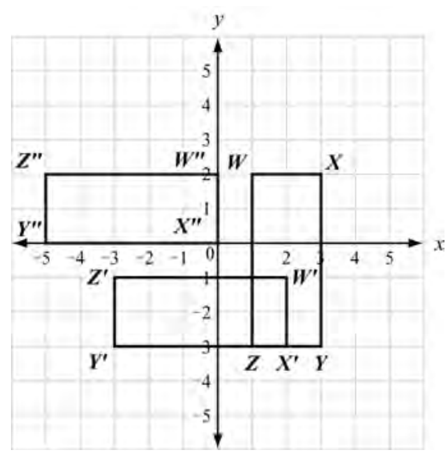
a. For a 90° clockwise rotation about the origin, use the rule $(x, y) \rightarrow (y, -x)$.

$W(1, 2) \rightarrow W'(2, -1)$
$X(3, 2) \rightarrow X'(2, -3)$
$Y(3, -3) \rightarrow Y'(-3, -3)$
$Z(1, -3) \rightarrow Z'(-3, -1)$



b. To translate rectangle $W'X'Y'Z'$ 2 units left and 3 units up, use the rule $(x, y) \rightarrow (x - 2, y + 3)$.

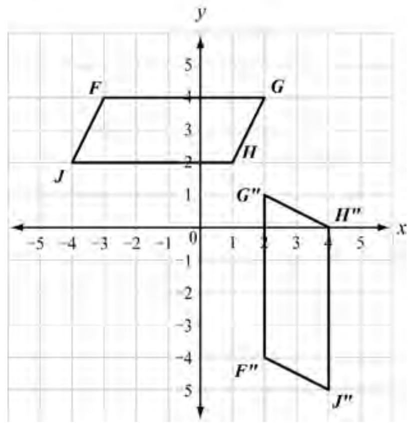
$W'(2, -1) \rightarrow W''(0, 2)$
$X'(2, -3) \rightarrow X''(0, 0)$
$Y'(-3, -3) \rightarrow Y''(-5, 0)$
$Z'(-3, -1) \rightarrow Z''(-5, 2)$



c. Rectangle $W''X''Y''Z''$ is the result of a rotation and a translation of rectangle *WXYZ*. These are both rigid transformations, so the shape and the size of the original figure are unchanged. All of the corresponding parts of *WXYZ* and $W''X''Y''Z''$ are congruent, so *WXYZ* and $W''X''Y''Z''$ are congruent.

EOCT Practice Items

1) Parallelogram *FGHJ* was translated 3 units down to form parallelogram $F'G'H'J'$. Parallelogram $F'G'H'J'$ was then rotated 90° counterclockwise about point G' to obtain parallelogram $F''G''H''J''$.

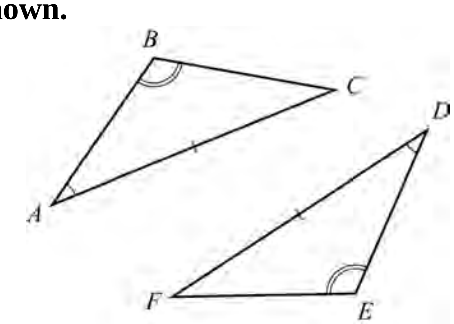


Which statement is true about parallelogram *FGHJ* and parallelogram $F''G''H''J''$?

A. The figures are both similar and congruent.
B. The figures are neither similar nor congruent.
C. The figures are similar but not congruent.
D. The figures are congruent but not similar.

[Key: A]

2) Consider the triangles shown.

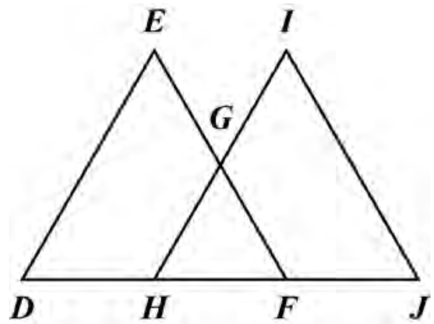


Which can be used to prove the triangles are congruent?

A. SSS
B. ASA
C. SAS
D. AAS

[Key: D]

3) In this diagram, $\overline{DE} \cong \overline{JI}$ and $\angle D \cong \angle J$.



Which additional information is sufficient to prove that $\triangle DEF$ is congruent to $\triangle JIH$?

A. $\overline{EF} \cong \overline{IH}$
B. $\overline{DH} \cong \overline{JF}$
C. $\overline{HG} \cong \overline{GI}$
D. $\overline{HF} \cong \overline{JF}$

[Key: B]

PROVE GEOMETRIC THEOREMS

KEY IDEAS

1. A ***two-column proof*** is a series of statements and reasons often displayed in a chart that works from given information to the statement that needs to be proven. Reasons can be given information, based on definitions, or based on postulates or theorems.

2. A ***paragraph proof*** also uses a series of statements and reasons that works from given information to the statement that needs to be proven, but the information is presented as running text in paragraph form.

3. It is important to plan a geometric proof logically. Think through what needs to be proven and decide how to get to that statement from the given information. Often a diagram or a flow chart will help to organize your thoughts.

4. An ***auxiliary line*** is a line drawn in a diagram that makes other figures such as congruent triangles or angles formed by a transversal. Many times an auxiliary line is needed to help complete a proof.

5. Once a theorem in geometry has been proven, that theorem can be used as a reason in future proofs.

6. Some important key ideas about lines and angles include:
    - ***Vertical Angle Theorem***: Vertical angles are congruent.
    - ***Alternate Interior Angles Theorem***: If two parallel lines are cut by a transversal, then alternate interior angles formed by the transversal are congruent.
    - ***Corresponding Angles Postulate***: If two parallel lines are cut by a transversal, then corresponding angles formed by the transversal are congruent.
    - Points on a perpendicular bisector of a line segment are equidistant from both of the segment's endpoints.

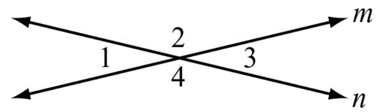
7. Some important key ideas about triangles include:
    - ***Triangle Angle-Sum Theorem***: The sum of the measures of the angles of a triangle is 180°.
    - ***Isosceles Triangle Theorem***: If two sides of a triangle are congruent, then the angles opposite those sides are also congruent.
    - ***Triangle Midsegment Theorem***: If a segment joins the midpoints of two sides of a triangle, then the segment is parallel to the third side and half its length.
    - The medians of a triangle meet at a point called the ***centroid***.

8. Some important key ideas about parallelograms include:

- Opposite sides are congruent and opposite angles are congruent.
- The diagonals of a parallelogram bisect each other.
- If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.
- A rectangle is a parallelogram with congruent diagonals.

REVIEW EXAMPLES

1) In this diagram, line *m* intersects line *n*.



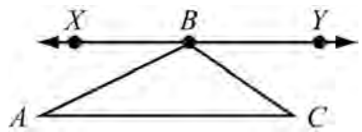
Write a two-column proof to show that vertical angles $\angle 1$ and $\angle 3$ are congruent.

Solution:

Construct a proof using intersecting lines.

Step	Statement	Justification
1	line m intersects line n	Given
2	$\angle 1$ and $\angle 2$ form a linear pair $\angle 2$ and $\angle 3$ form a linear pair	Definition of a linear pair
3	$m\angle 1 + m\angle 2 = 180°$ $m\angle 2 + m\angle 3 = 180°$	Angles that form a linear pair have measures that sum to 180°
4	$m\angle 1 + m\angle 2 = m\angle 2 + m\angle 3$	Substitution
5	$m\angle 1 = m\angle 3$	Subtraction Property of Equality
6	$\angle 1 \cong \angle 3$	Definition of congruent angles

2) In this diagram, $\overleftrightarrow{XY}$ is parallel to $\overline{AC}$, and point B lies on $\overleftrightarrow{XY}$.

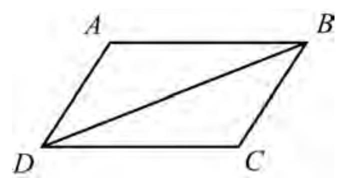


Write a paragraph to prove that the sum of the angles in a triangle is 180°.

Solution:

$\overline{AC}$ and $\overleftrightarrow{XY}$ are parallel, so $\overline{AB}$ is a transversal. The alternate interior angles formed by the transversal are congruent. So, $m\angle A = m\angle ABX$. Similarly, $\overline{BC}$ is a transversal, so $m\angle C = m\angle CBY$. The sum of the angle measures that make a straight line is 180°. So, $m\angle ABX + m\angle ABC + m\angle CBY = 180°$. Now, substitute $m\angle A$ for $m\angle ABX$ and $m\angle C$ for $m\angle CBY$ to get $m\angle A + m\angle ABC + m\angle C = 180°$.

3) In this diagram, $ABCD$ is a parallelogram and $\overline{BD}$ is a diagonal.



Write a two-column proof to show that $\overline{AB}$ and $\overline{CD}$ are congruent.

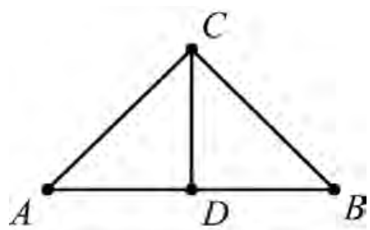
Solution:

Construct a proof using properties of the parallelogram and its diagonal.

Step	Statement	Justification
1	$ABCD$ is a parallelogram	Given
2	$\overline{BD}$ is a diagonal	Given
3	$\overline{AB}$ is parallel to $\overline{DC}$ $\overline{AD}$ is parallel to $\overline{BC}$	Definition of parallelogram
4	$\angle ABD \cong \angle CDB$ $\angle DBC \cong \angle BDA$	Alternate interior angles are congruent.
5	$\overline{BD} \cong \overline{BD}$	Reflexive Property of Congruence
6	$\triangle ADB \cong \triangle CBD$	ASA
7	$\overline{AB} \cong \overline{CD}$	CPCTC

EOCT Practice Items

1) In this diagram, $\overline{CD}$ is the perpendicular bisector of $\overline{AB}$. The two-column proof shows that $\overline{AC}$ is congruent to $\overline{BC}$.



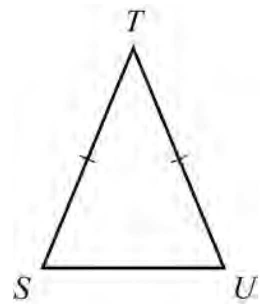
Step	Statement	Justification
1	$\overline{CD}$ is the perpendicular bisector of $\overline{AB}$	Given
2	$\overline{AD} \cong \overline{BD}$	Definition of bisector
3	$\overline{CD} \cong \overline{CD}$	Reflexive Property of Congruence
4	$\angle ADC$ and $\angle BDC$ are right angles	Definition of perpendicular lines
5	$\angle ADC \cong \angle BDC$	All right angles are congruent
6	$\triangle ADC \cong \triangle BDC$	**?**
7	$\overline{AC} \cong \overline{BC}$	CPCTC

Which theorem would justify Step 6?

A. AAS
B. ASA
C. SAS
D. SSS

[Key: C]

2) In this diagram, *STU* is an isosceles triangle where $\overline{ST}$ is congruent to $\overline{UT}$. The paragraph proof shows that $\angle S$ is congruent to $\angle U$.



It is given that $\overline{ST}$ is congruent to $\overline{UT}$. Draw $\overline{TV}$ that bisects $\angle T$. By the definition of an angle bisector, $\angle STV$ is congruent to $\angle UTV$. By the Reflexive Property, $\overline{TV}$ is congruent to $\overline{TV}$. Triangle *STV* is congruent to triangle *UTV* by SAS. $\angle S$ is congruent to $\angle U$ by ______?______.

Which step is missing in the proof?

A. CPCTC
B. Reflexive Property of Congruence
C. Definition of right angles
D. Angle Congruence Postulate

[Key: A]

MAKE GEOMETRIC CONSTRUCTIONS

KEY IDEAS

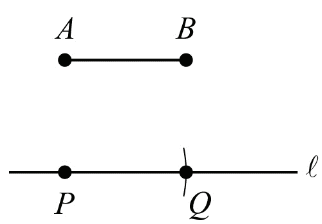
1. To ***copy a segment***, follow the steps given:

Given: $\overline{AB}$

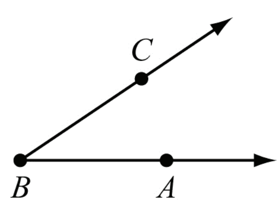
Construct: $\overline{PQ}$ congruent to $\overline{AB}$

Procedure:

1. Use a straightedge to draw a line, *l*.
2. Choose a point on line *l* and label it point *P*.
3. Place the compass point on point *A*.
4. Adjust the compass width to the length of $\overline{AB}$.
5. Without changing the compass, place the compass point on point *P* and draw an arc intersecting line *l*. Label the point of intersection as point *Q*.
6. $\overline{PQ} \cong \overline{AB}$.



2. To ***copy an angle***, follow the steps given:



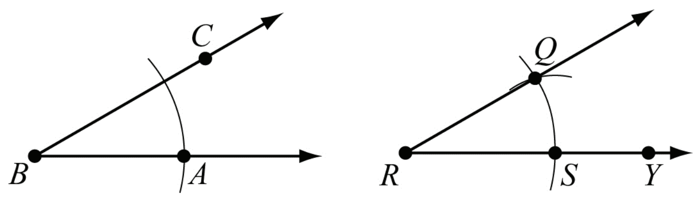
Given: $\angle ABC$

Construct: $\angle QRY$ congruent to $\angle ABC$

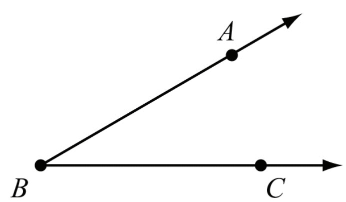
Procedure:

1. Draw a point *R* that will be the vertex of the new angle.
2. From point *R*, use a straightedge to draw $\overrightarrow{RY}$, which will become one side of the new angle.

3. Place the compass point on vertex B and draw an arc through point A.
4. Without changing the compass, place the compass point on point R, draw an arc intersecting $\overrightarrow{RY}$, and label the point of intersection point S.
5. Place the compass point on point A and adjust its width to where the arc intersects $\overrightarrow{BC}$.
6. Without changing the compass width, place the compass point on point S and draw another arc across the first arc. Label the point where both arcs intersect as point Q.
7. Use a straightedge to draw $\overrightarrow{RQ}$.
8. $\angle QRY \cong \angle ABC$



3. To ***bisect an angle***, follow the steps given:



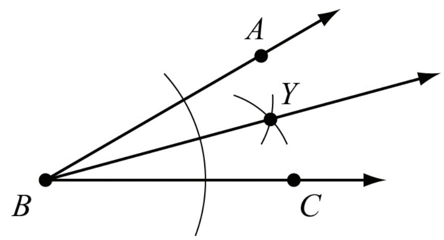
Given: $\angle ABC$

Construct: $\overrightarrow{BY}$, the bisector of $\angle ABC$

Procedure:

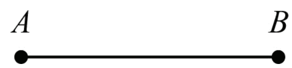
1. Place the compass point on vertex B.
2. Open the compass and draw an arc that crosses both sides of the angle.
3. Set the compass width to more than half the distance from point B to where the arc crosses $\overrightarrow{BA}$. Place the compass point where the arc crosses $\overrightarrow{BA}$ and draw an arc in the angle's interior.
4. Without changing the compass width, place the compass point where the arc crosses $\overrightarrow{BC}$ and draw an arc so that it crosses the previous arc. Label the intersection point Y.
5. Using a straightedge, draw a ray from vertex B through point Y.

6. $\angle ABY \cong \angle YBC$, so $\overrightarrow{BY}$ is the bisector of $\angle ABC$.



4. To ***construct a perpendicular bisector of a line segment***, follow the steps given:

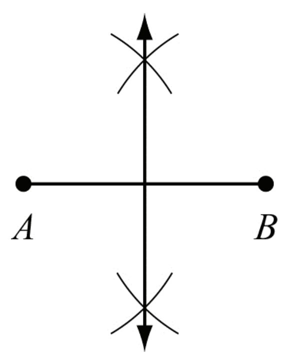
Given: $\overline{AB}$



Construct: The perpendicular bisector of $\overline{AB}$

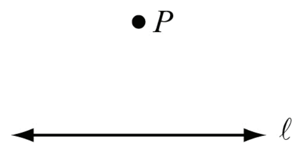
Procedure:

1. Adjust the compass to a width greater than half the length of $\overline{AB}$.
2. Place the compass on point A and draw an arc passing above $\overline{AB}$ and an arc passing below $\overline{AB}$.
3. Without changing the compass width, place the compass on point B and draw an arc passing above and below $\overline{AB}$.
4. Use a straightedge to draw a line through the points of intersection of these arcs.
5. The segment is the perpendicular bisector of $\overline{AB}$.



Note: To bisect $\overline{AB}$, follow the same steps listed above to construct the perpendicular bisector. The point where the perpendicular bisector intersects $\overline{AB}$ is the midpoint of $\overline{AB}$.

5. To ***construct a line perpendicular to a given line through a point not on the line***, follow the steps given:

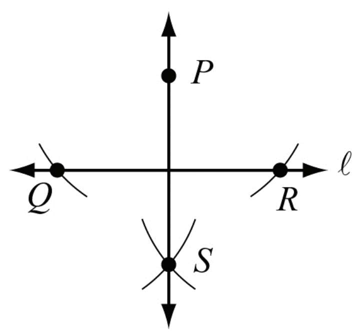


Given: Line *l* and point *P* that is not on line *l*

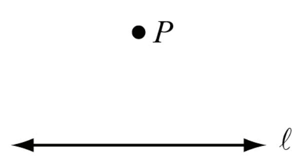
Construct: The line perpendicular to line *l* through point *P*

Procedure:

1. Place the compass point on point *P*.
2. Open the compass to a distance that is wide enough to draw two arcs across line *l*, one on each side of point *P*. Label these points *Q* and *R*.
3. From points *Q* and *R*, draw arcs on the opposite side of line *l* from point *P* so that the arcs intersect. Label the intersection point *S*.
4. Using a straightedge, draw $\overleftrightarrow{PS}$.
5. $\overline{PS} \perp \overline{QR}$.



6. To ***construct a line parallel to a given line through a point not on the line***, follow the steps given:

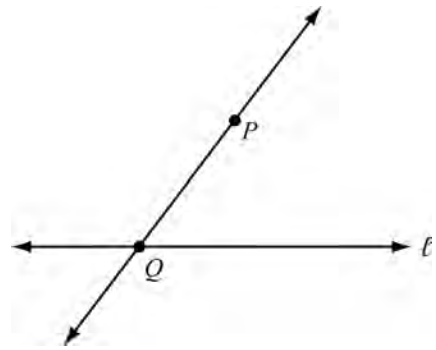


Given: Line *l* and point *P* that is not on line *l*

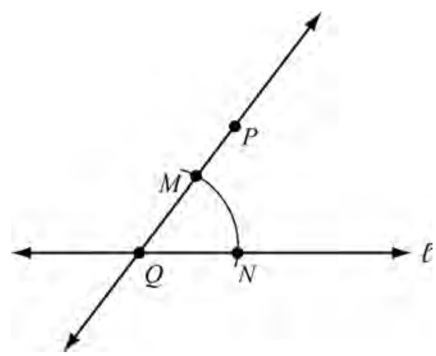
Construct: The line parallel to line *l* through point *P*

Procedure:

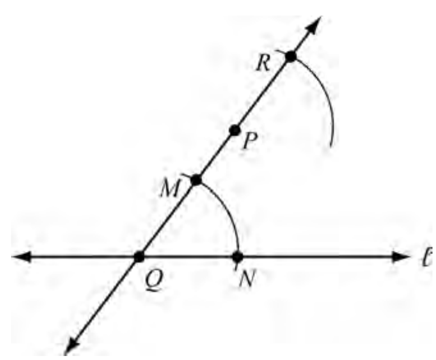
1. Draw a transversal line through point P crossing line l at a point. Label the point of intersection Q.



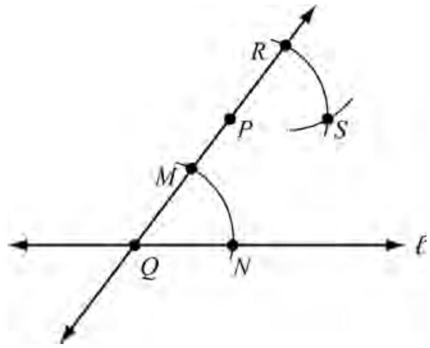
2. Open the compass to a width about half the distance from points P to Q. Place the compass point on point Q and draw an arc that intersects both lines. Label the intersection of the arc and $\overline{PQ}$ as point M and the intersection of the arc and l as point N.



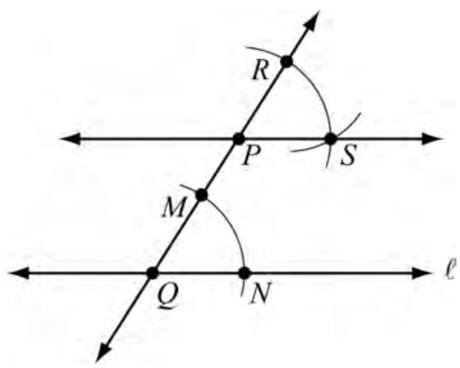
3. Without changing the compass width, place the compass point on point P and draw an arc that crosses $\overline{PQ}$ above point P. Note that this arc must have the same orientation as the arc drawn from points M to N. Label the point R.



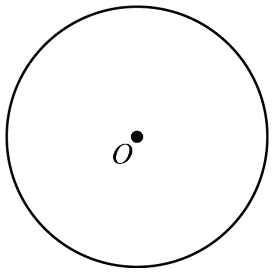
4. Set the compass width to the distance from points M to N.
5. Place the compass point on point R and draw an arc that crosses the upper arc. Label the point of intersection S.



6. Using a straightedge, draw a line through points P and S.
7. $\overleftrightarrow{PS} \parallel l$



7. To ***construct an equilateral triangle inscribed in a circle***, follow the steps given:

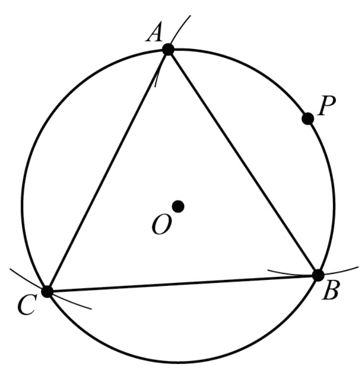


Given: Circle O

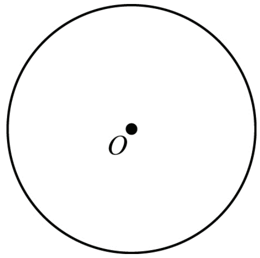
Construct: Equilateral $\triangle ABC$ inscribed in circle O

Procedure:

1. Mark a point anywhere on the circle and label it point P.
2. Open the compass to the radius of circle O.
3. Place the compass point on point P and draw an arc that intersects the circle at two points. Label the points A and B.
4. Using a straightedge, draw $\overline{AB}$.
5. Open the compass to the length of $\overline{AB}$.
6. Place the compass point on A. Draw an arc from point A that intersects the circle. Label this point C.
7. Using a straightedge, draw $\overline{AC}$ and $\overline{BC}$.
8. Equilateral $\triangle ABC$ is inscribed in circle O.



8. To ***construct a square inscribed in a circle***, follow the steps given:

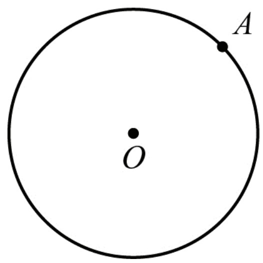


Given: Circle *O*

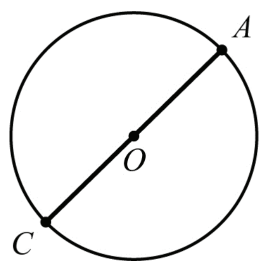
Construct: Square *ABCD* inscribed in circle *O*

Procedure:

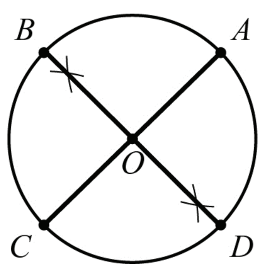
1. Mark a point anywhere on the circle and label it point *A*.



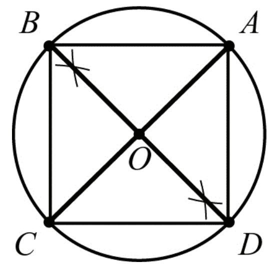
2. Using a straightedge, draw a diameter from point *A*. Label the other endpoint of the diameter as point *C*. This is diameter $\overline{AC}$.



3. Construct a perpendicular bisector to $\overline{AC}$ through the center of circle *O*. Label the points where it intersects the circle as point *B* and point *D*.

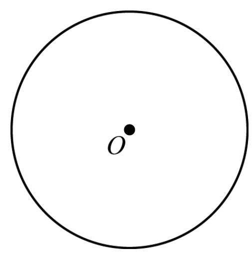


4. Using a straightedge, draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$.



5. Square *ABCD* is inscribed in circle *O*.

9. To ***construct a regular hexagon inscribed in a circle***, follow the steps given:

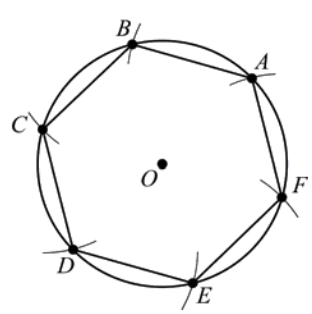


Given: Circle *O*

Construct: Regular hexagon *ABCDEF* inscribed in circle *O*

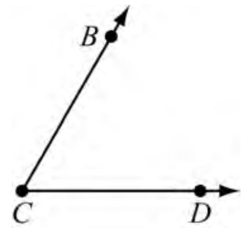
Procedure:

1. Mark a point anywhere on the circle and label it point *A*.
2. Open the compass to the radius of circle *O*.
3. Place the compass point on point *A* and draw an arc across the circle. Label this point *B*.
4. Without changing the width of the compass, place the compass point on *B* and draw another arc across the circle. Label this point *C*.
5. Repeat this process from point *C* to a point *D*, from point *D* to a point *E*, and from point *E* to a point *F*.
6. Use a straightedge to draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{AF}$.
7. Regular hexagon *ABCDEF* is inscribed in circle *O*.



REVIEW EXAMPLES

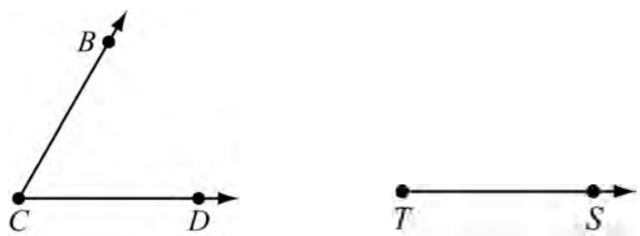
1) Allan drew angle *BCD*.



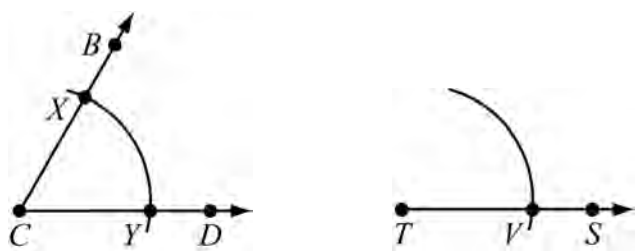
a. Copy angle *BCD*. List the steps you used to copy the angle. Label the copied angle *RTS*.
b. Without measuring the angles, how can you show they are congruent to one another?

Solution:

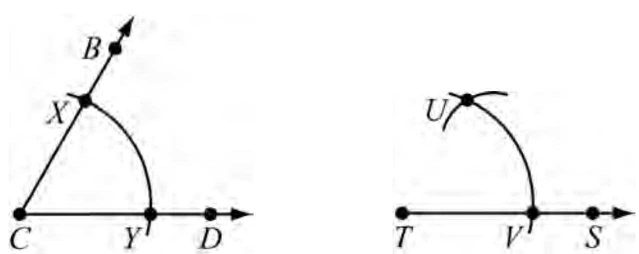
a. Draw point *T*. Draw $\overrightarrow{TS}$.



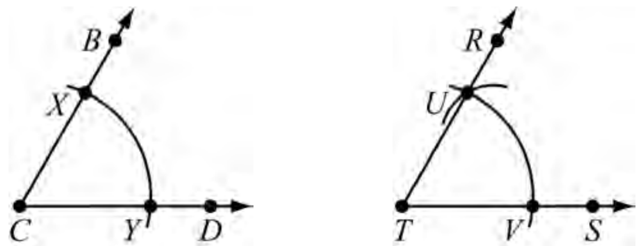
Place the point of a compass on point *C*. Draw an arc. Label the intersection points *X* and *Y*. Keep the compass width the same, and place the point of the compass on point *T*. Draw an arc and label the intersection point *V*.



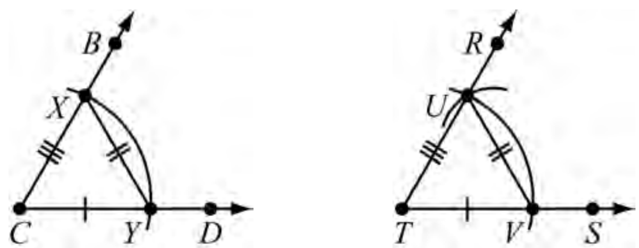
Place the point of the compass on point *Y* and adjust the width to point *X*. Then place the point of the compass on point *V* and draw an arc that intersects the first arc. Label the intersection point *U*.



Draw $\overrightarrow{TU}$ and point *R* on $\overrightarrow{TU}$. Angle *BCD* has now been copied to form angle *RTS*.



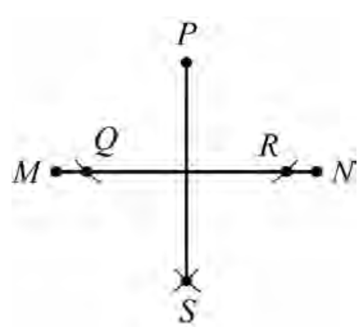
b. Connect points *X* and *Y* and points *U* and *V* to form $\triangle XCY$ and $\triangle UTV$. $\overline{CY}$ and $\overline{TV}$, $\overline{XY}$ and $\overline{UV}$, and $\overline{CX}$ and $\overline{TU}$ are congruent because they were drawn with the same compass width. So, $\triangle XCY \cong \triangle UTV$ by SSS, and $\angle C \cong \angle T$ because congruent parts of congruent triangles are congruent.



2) Construct a line segment perpendicular to $\overline{MN}$ from a point not on $\overline{MN}$. Explain the steps you used to make your construction.

Solution:

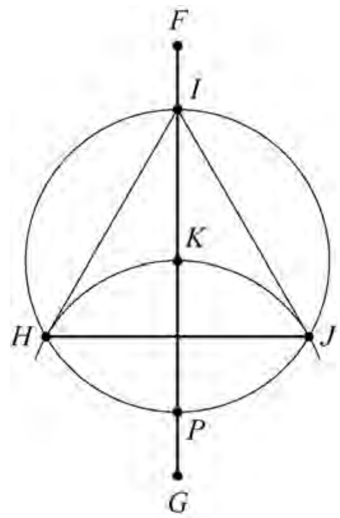
Draw a point *P* that is not on $\overline{MN}$. Place the compass point on point *P*. Draw an arc that intersects $\overline{MN}$ at two points. Label the intersections points *Q* and *R*. Without changing the width of the compass, place the compass on point *Q* and draw an arc under $\overline{MN}$. Place the compass on point *R* and draw another arc under $\overline{MN}$. Label the intersection point *S*. Draw $\overline{PS}$. Segment *PS* is perpendicular to and bisects $\overline{MN}$.



3) Construct equilateral $\triangle HIJ$ inscribed in circle *K*. Explain the steps you used to make your construction.

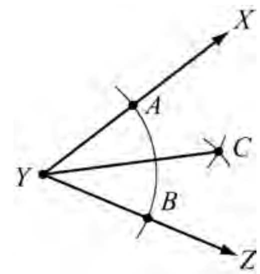
Solution:

(This is an alternate method from the method shown in Key Idea 7.) Draw circle *K*. Draw segment $\overline{FG}$ through the center of circle *K*. Label the intersection points *I* and *P*. Using the compass setting you used when drawing the circle, place a compass on point *P* and draw an arc passing through point *K*. Label the intersection points at either side of the circle points *H* and *J*. Draw $\overline{HJ}$, $\overline{IJ}$, and $\overline{HI}$. Triangle *HIJ* is an equilateral triangle inscribed in circle *K*.



EOCT Practice Items

1) Consider the construction of the angle bisector shown.



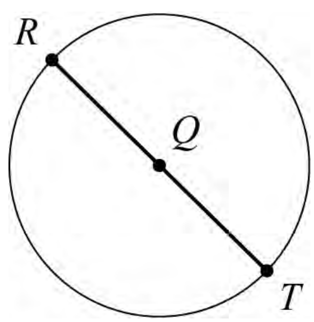
Which could have been the first step in creating this construction?

A. Place the compass point on point A and draw an arc inside $\angle Y$.
B. Place the compass point on point B and draw an arc inside $\angle Y$.
C. Place the compass point on vertex Y and draw an arc that intersects $\overline{YX}$ and $\overline{YZ}$.
D. Place the compass point on vertex Y and draw an arc that intersects point C.

[Key: C]

2) Consider the beginning of a construction of a square inscribed in circle *Q*.

Step 1: Label point *R* on circle *Q*.
Step 2: Draw a diameter through *R* and *Q*.
Step 3: Label the intersection on the circle point *T*.



What is the next step in this construction?

A. Draw radius $\overline{SQ}$.
B. Label point S on circle Q.
C. Construct a line segment parallel to $\overline{RT}$.
D. Construct the perpendicular bisector of $\overline{RT}$.

[Key: D]

Unit 2: Right Triangle Trigonometry

This unit investigates the properties of right triangles. The trigonometric ratios sine, cosine, and tangent along with the Pythagorean Theorem are used to solve right triangles in applied problems. The relationship between the sine and cosine of complementary angles is identified.

KEY STANDARDS

Define trigonometric ratios and solve problems involving right triangles

MCC9-12.G.SRT.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

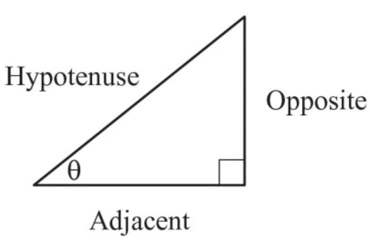
MCC9-12.G.SRT.7 Explain and use the relationship between the sine and cosine of complementary angles.

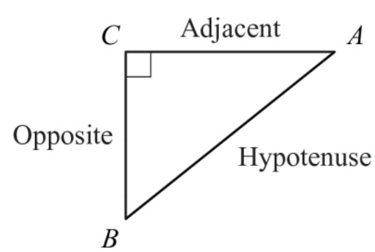
MCC9-12.G.SRT.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

RIGHT TRIANGLE RELATIONSHIPS

KEY IDEAS

1. The trigonometric ratios ***sine***, ***cosine***, and ***tangent*** are defined as ratios of the lengths of the sides in a right triangle with a given acute angle measure. These terms are usually seen abbreviated as ***sin***, ***cos***, and ***tan***.





- $\sin\theta = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$.
- $\cos\theta = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.
- $\tan\theta = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.

- $\sin A = \dfrac{\text{length of opposite side}}{\text{length of hypotenuse}}$.
- $\cos A = \dfrac{\text{length of adjacent side}}{\text{length of hypotenuse}}$.
- $\tan A = \dfrac{\text{length of opposite side}}{\text{length of adjacent side}}$.

2. The two acute angles of any right triangle are complementary. As a result, if angles P and Q are complementary, $\sin P = \cos Q$ and $\sin Q = \cos P$.

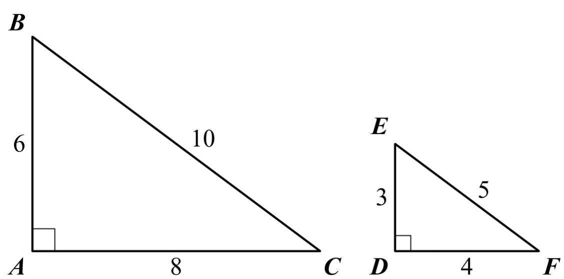
3. When solving problems with right triangles, you can use both trigonometric ratios and the Pythagorean Theorem ($a^2 + b^2 = c^2$). There may be more than one way to solve the problem, so analyze the given information to help decide which method is the most efficient.

Important Tip

The tangent of angle A is also equivalent to $\dfrac{\sin A}{\cos A}$.

REVIEW EXAMPLES

1) Triangles ABC and DEF are similar.



a. Find the ratio of the side opposite angle B to the hypotenuse in $\triangle ABC$.

b. What angle in $\triangle DEF$ corresponds to angle B?

c. Find the ratio of the side opposite angle E to the hypotenuse in $\triangle DEF$.

d. How does the ratio in part (a) compare to the ratio in part (c)?

e. Which trigonometric ratio does this represent?

Solution:

a. $\overline{AC}$ is opposite angle B. $\overline{BC}$ is the hypotenuse. The ratio of the side opposite angle B to the hypotenuse in $\triangle ABC$ is $\frac{8}{10} = \frac{4}{5}$.

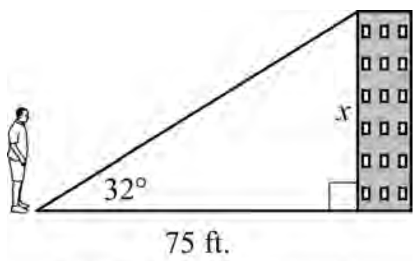
b. Angle E in $\triangle DEF$ corresponds to angle B in $\triangle ABC$.

c. $\overline{DF}$ is opposite angle E. $\overline{EF}$ is the hypotenuse. The ratio of the side opposite angle E to the hypotenuse in $\triangle DEF$ is $\frac{4}{5}$.

d. The ratios are the same.

e. This represents sin B and sin E, because both are the ratio $\frac{\text{opposite}}{\text{hypotenuse}}$.

2) Ricardo is standing 75 feet away from the base of a building. The angle of elevation from the ground where Ricardo is standing to the top of the building is 32°.



Note: Figure not drawn to scale.

What is x, the height of the building, to the nearest tenth of a foot?

$$\begin{bmatrix} \sin 32^\circ = 0.5299 \\ \cos 32^\circ = 0.8480 \\ \tan 32^\circ = 0.6249 \end{bmatrix}$$

Solution:

You want to know the length of the side opposite of the 32° angle, and you know the length of the side adjacent to the 32° angle. So, use the tangent ratio. Substitute x for the opposite side, 75 for the adjacent side, and 32° for the angle measure. Then solve.

$$\tan 32^\circ = \frac{x}{75}$$

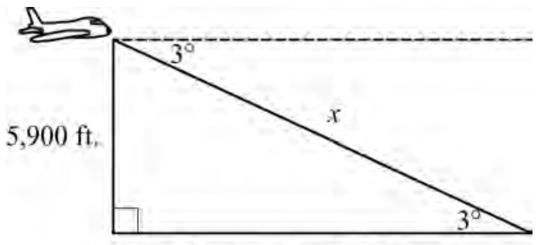
$$75 \tan 32^\circ = x$$

$$75 \bullet 0.6249 \approx x$$

$$46.9 \approx x$$

The building is about 46.9 feet tall.

3) An airplane is at an altitude of 5,900 feet. The airplane descends at an angle of 3°.



Note: Figure not drawn to scale.

About how far will the airplane travel in the air until it reaches the ground?

$$\begin{bmatrix} \sin 3^\circ = 0.0523 \\ \cos 3^\circ = 0.9986 \\ \tan 3^\circ = 0.0524 \end{bmatrix}$$

Solution:

Use sin 3° to find the distance the airplane will travel until it reaches the ground, *x*. Substitute *x* for the hypotenuse, 5,900 for the opposite side, and 3° for the angle measure. Then solve.

$$\sin 3^\circ = \frac{5{,}900}{x}$$

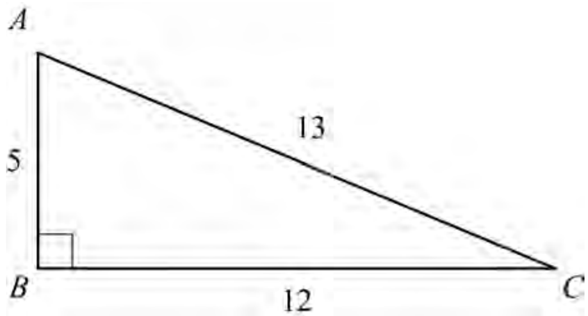
$$x = \frac{5{,}900}{\sin 3^\circ}$$

$$x \approx \frac{5{,}900}{0.0523}$$

$$x \approx 112{,}811$$

The airplane will travel about 113,000 feet until it reaches the ground.

4) Triangle ABC is a right triangle.



What is the best approximation for $m\angle C$?

$$\begin{bmatrix} \sin 67.4^\circ \approx 0.923 \\ \cos 22.6^\circ \approx 0.923 \\ \tan 42.7^\circ \approx 0.923 \end{bmatrix}$$

Solution:

Find the trigonometric ratios for angle C.

$$\sin C = \frac{5}{13} \approx 0.385$$

$$\cos C = \frac{12}{13} \approx 0.923$$

$$\tan C = \frac{5}{12} \approx 0.417$$

Using the table, $\cos 22.6^\circ \approx 0.923$, so $m\angle C \approx 22.6^\circ$.

EOCT Practice Items

1) In right triangle *ABC*, angle *A* and angle *B* are complementary angles. The value of $\cos A$ is $\frac{5}{13}$. What is the value of $\sin B$?

A. $\frac{5}{13}$

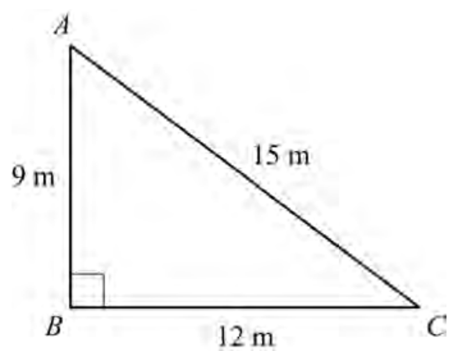
B. $\frac{12}{13}$

C. $\frac{13}{12}$

D. $\frac{13}{5}$

[Key: A]

2) Triangle *ABC* is given below.



What is the value of $\cos A$?

A. $\frac{3}{5}$

B. $\frac{3}{4}$

C. $\frac{4}{5}$

D. $\frac{5}{3}$

[Key: A]

3) In right triangle *HJK*, $\angle J$ is a right angle and $\tan\angle H = 1$. Which statement about triangle *HJK* must be true?

A. $\sin\angle H = \frac{1}{2}$

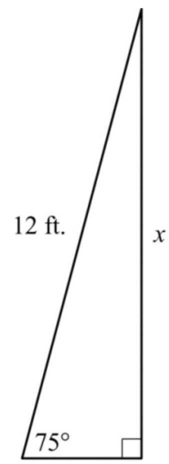
B. $\sin\angle H = 1$

C. $\sin\angle H = \cos\angle H$

D. $\sin\angle H = \frac{1}{\cos\angle H}$

[Key: C]

4) A 12-foot ladder is leaning against a building at a 75° angle with the ground.



Which can be used to find how high the ladder reaches up the side of the building?

A. $\sin 75^\circ = \frac{12}{x}$

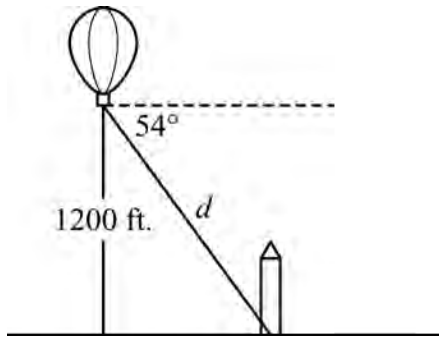
B. $\tan 75^\circ = \frac{12}{x}$

C. $\cos 75^\circ = \frac{x}{12}$

D. $\sin 75^\circ = \frac{x}{12}$

[Key: D]

5) A hot air balloon is 1200 feet above the ground. The angle of depression from the basket of the hot-air balloon to the base of a monument is 54°.



Which equation can be used to find the distance, *d*, in feet, from the basket of the hot-air balloon to the base of the monument?

A. $\sin 54° = \dfrac{d}{1200}$

B. $\sin 54° = \dfrac{1200}{d}$

C. $\cos 54° = \dfrac{d}{1200}$

D. $\cos 54° = \dfrac{1200}{d}$

[Key: B]

Unit 3: Circles and Volume

This unit investigates the properties of circles and addresses finding the volume of solids. Properties of circles are used to solve problems involving arcs, angles, sectors, chords, tangents, and secants. Volume formulas are derived and used to calculate the volumes of cylinders, pyramids, cones, and spheres.

KEY STANDARDS

Understand and apply theorems about circles

MCC9-12.G.C.1 Prove that all circles are similar.

MCC9-12.G.C.2 Identify and describe relationships among inscribed angles, radii, and chords. Include the relationship between central, inscribed, and circumscribed angles; inscribed angles on a diameter are right angles; the radius of a circle is perpendicular to the tangent where the radius intersects the circle.

MCC9-12.G.C.3 Construct the inscribed and circumscribed circles of a triangle, and prove properties of angles for a quadrilateral inscribed in a circle.

Find arc lengths and areas of sectors of circles

MCC9-12.G.C.5 Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

Explain volume formulas and use them to solve problems

MCC9-12.G.GMD.1 Give an informal argument for the formulas for the circumference of a circle, area of a circle, volume of a cylinder, pyramid, and cone. Use dissection arguments, Cavalieri's principle, and informal limit arguments.

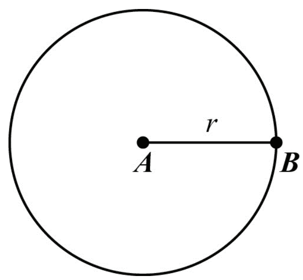
MCC9-12.G.GMD.3 Use volume formulas for cylinders, pyramids, cones, and spheres to solve problems.★

UNDERSTAND AND APPLY THEOREMS ABOUT CIRCLES

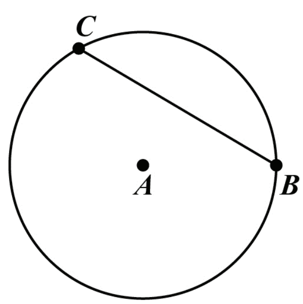
KEY IDEAS

1. A ***circle*** is the set of points in a plane equidistant from a given point, which is the center of the circle. All circles are similar.

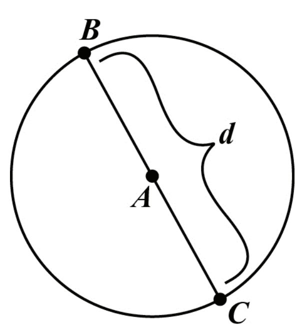
2. A ***radius*** is a line segment from the center of a circle to any point on the circle. The word radius is also used to describe the length, r, of the segment. $\overline{AB}$ is a radius of circle A.



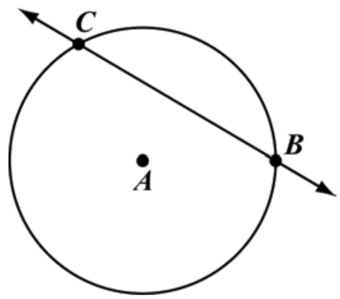
3. A ***chord*** is a line segment whose endpoints are on a circle. $\overline{BC}$ is a chord of circle A.



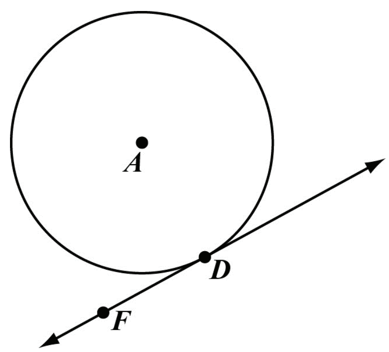
4. A ***diameter*** is a chord that passes through the center of a circle. The word diameter is also used to describe the length, d, of the segment. $\overline{BC}$ is a diameter of circle A.



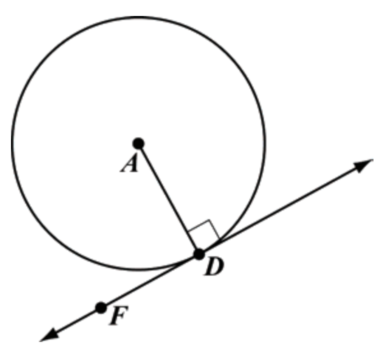
5. A ***secant line*** is a line that is in the plane of a circle and intersects the circle at exactly two points. Every chord lies on a secant line. $\overleftrightarrow{BC}$ is a secant line of circle *A*.



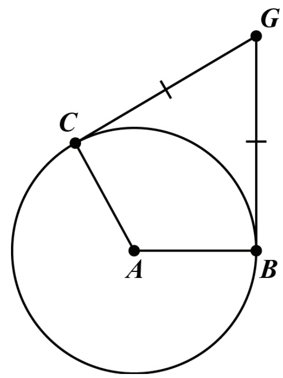
6. A ***tangent line*** is a line that is in the plane of a circle and intersects the circle at only one point, the ***point of tangency***. $\overleftrightarrow{DF}$ is tangent to circle *A* at the point of tangency, point *D*.



7. If a line is tangent to a circle, the line is perpendicular to the radius drawn to the point of tangency. $\overleftrightarrow{DF}$ is tangent to circle *A* at point *D*, so $\overline{AD} \perp \overline{DF}$.

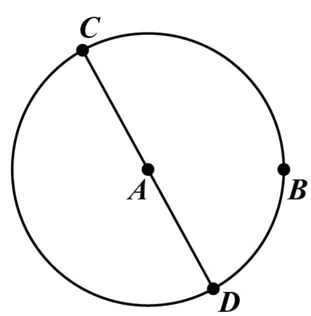


8. Tangent segments drawn from the same point are congruent. In circle A, $\overline{CG} \cong \overline{BG}$.

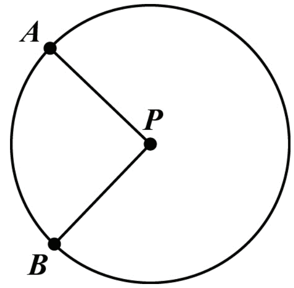


9. ***Circumference*** is the distance around a circle. The formula for circumference C of a circle is $C = \pi d$, where d is the diameter of the circle. The formula is also written as $C = 2\pi r$, where r is the length of the radius of the circle. π is the ratio of circumference to diameter of any circle.

10. An ***arc*** is a part of the circumference of a circle. A ***minor arc*** has a measure less than 180°. Minor arcs are written using two points on a circle. A ***semicircle*** is an arc that measures exactly 180°. Semicircles are written using three points on a circle. This is done to show which half of the circle is being described. A ***major arc*** has a measure greater than 180°. Major arcs are written with three points to distinguish them from the corresponding minor arc. In circle A, $\overset{\frown}{CB}$ is a minor arc, $\overset{\frown}{CBD}$ is a semicircle, and $\overset{\frown}{CDB}$ is a major arc.

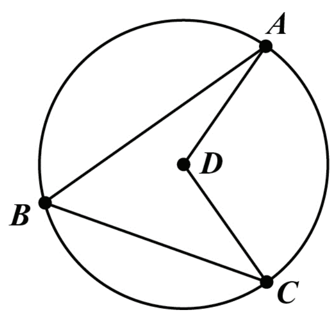


11. A ***central angle*** is an angle whose vertex is at the center of a circle and whose sides are radii of the circle. The measure of a central angle of a circle is equal to the measure of the intercepted arc. $\angle APB$ is a central angle for circle P and $\overparen{AB}$ is the intercepted arc.



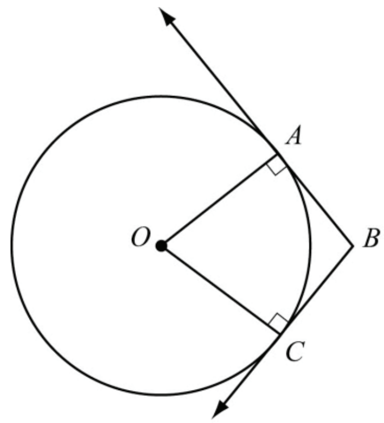
$$m\angle APB = m\overparen{AB}$$

12. An ***inscribed angle*** is an angle whose vertex is on a circle and whose sides are chords of the circle. The measure of an angle inscribed in a circle is half the measure of the intercepted arc. For circle D, $\angle ABC$ is an inscribed angle and $\overparen{AC}$ is the intercepted arc.



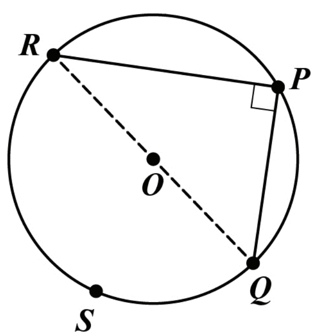
$$m\angle ABC = \frac{1}{2}m\overparen{AC} = \frac{1}{2}m\angle ADC$$
$$m\angle ADC = m\overparen{AC} = 2(m\angle ABC)$$

13. A ***circumscribed angle*** is an angle formed by two rays that are each tangent to a circle. These rays are perpendicular to radii of the circle. In circle *O*, the measure of the circumscribed angle is equal to 180° minus the measure of the central angle that forms the intercepted arc. The measure of the circumscribed angle can also be found by using the measures of the intercepted arcs [see Key Idea 18].



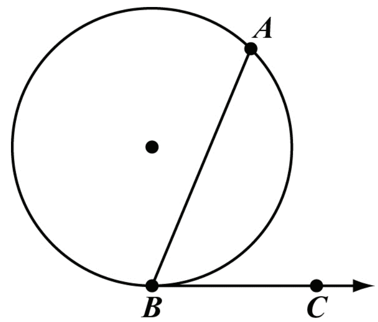
$$m\angle ABC = 180° - m\angle AOC$$

14. When an inscribed angle intercepts a semicircle, the inscribed angle has a measure of 90º. For circle *O*, $\angle RPQ$ intercepts semicircle $\overparen{RSQ}$ as shown.



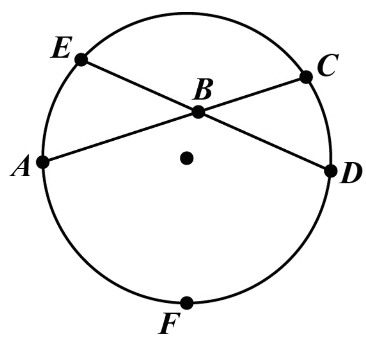
$$m\angle RPQ = \frac{1}{2}\left(m\overparen{RSQ}\right) = \frac{1}{2}(180°) = 90°$$

15. The measure of an angle formed by a tangent and a chord with its vertex on the circle is half the measure of the intercepted arc. $\overline{AB}$ is a chord for the circle and $\overrightarrow{BC}$ is tangent to the circle at point *B*. So, $\angle ABC$ is formed by a tangent and a chord.



$$m\angle ABC = \frac{1}{2}\left(m\overset{\frown}{AB}\right)$$

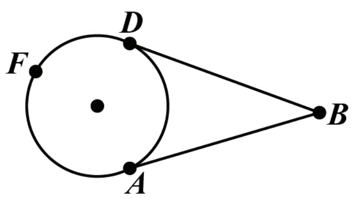
16. When two chords intersect inside a circle, two pairs of vertical angles are formed. The measure of any one of the angles is half the sum of the measures of the arcs intercepted by the pair of vertical angles.

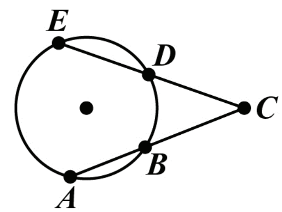


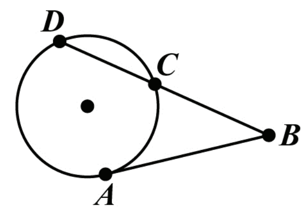
$$m\angle ABE = \frac{1}{2}\left(m\overset{\frown}{AE} + m\overset{\frown}{CD}\right)$$
$$m\angle ABD = \frac{1}{2}\left(m\overset{\frown}{AFD} + m\overset{\frown}{EC}\right)$$

17. When two chords intersect inside a circle, the product of the lengths of the segments of one chord is equal to the product of the lengths of the segments of the other chord. In the circle in Key Idea 15, $AB \bullet BC = EB \bullet BD$.

18. Angles outside a circle can be formed by the intersection of two tangents (circumscribed angle), two secants, or a secant and a tangent. For all three situations, the measure of the angle is half the difference of the measure of the larger intercepted arc and the measure of the smaller intercepted arc.

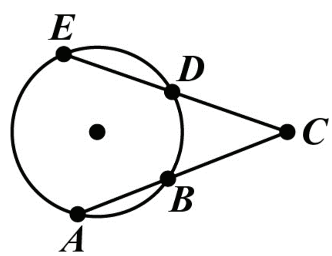




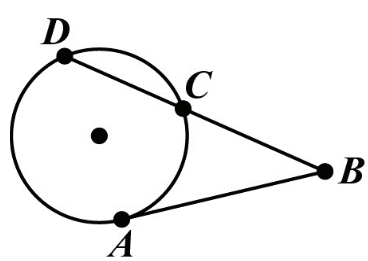


$m\angle ABD = \frac{1}{2}\left(m\widehat{AFD} - m\widehat{AD}\right)$ $m\angle ACE = \frac{1}{2}\left(m\widehat{AE} - m\widehat{BD}\right)$ $m\angle ABD = \frac{1}{2}\left(m\widehat{AD} - m\widehat{AC}\right)$

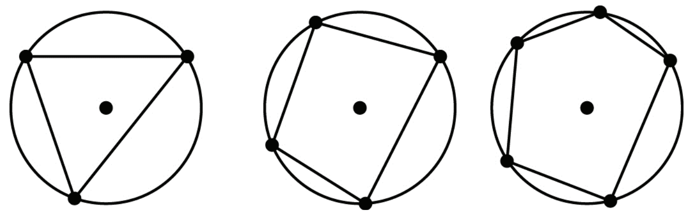
19. When two secant segments intersect outside a circle, part of each secant segment is a segment formed outside the circle. The product of the length of one secant segment and the length of the segment formed outside the circle is equal to the product of the length of the other secant segment and the length of the segment formed outside the circle. For secant segments $\overline{EC}$ and $\overline{AC}$ shown, $EC \bullet DC = AC \bullet BC$.



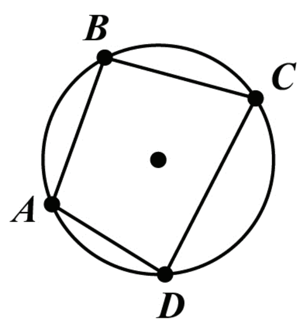
20. When a secant segment and a tangent segment intersect outside a circle, the product of the length of the secant segment and the length of the segment formed outside the circle is equal to the square of the length of the tangent segment. For secant segment $\overline{DB}$ and tangent segment $\overline{AB}$ shown, $DB \bullet CB = AB^2$.



21. An ***inscribed polygon*** is a polygon whose vertices all lie on a circle. This diagram shows a triangle, a quadrilateral, and a pentagon inscribed in a circle.

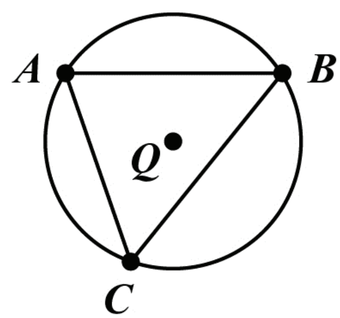


22. In a quadrilateral inscribed in a circle, the opposite angles are supplementary.

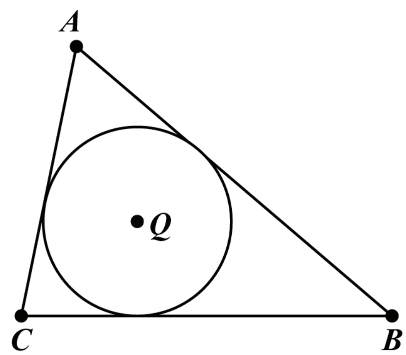


$$m\angle ABC + m\angle ADC = 180^\circ$$
$$m\angle BCD + m\angle BAD = 180^\circ$$

23. When a triangle is inscribed in a circle, the center of the circle is the ***circumcenter*** of the triangle. The circumcenter is equidistant from the vertices of the triangle. Triangle *ABC* is inscribed in circle *Q*, and point *Q* is the circumcenter of the triangle.

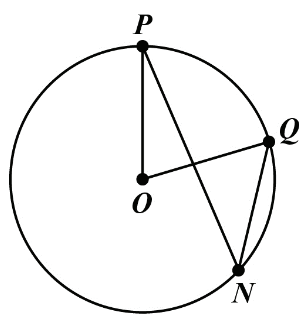


24. An ***inscribed circle*** is a circle enclosed in a polygon, where every side of the polygon is tangent to the circle. Specifically, when a circle is inscribed in a triangle, the center of the circle is the ***incenter*** of the triangle. The incenter is equidistant from the sides of the triangle. Circle Q is inscribed in triangle ABC, and point Q is the incenter of the triangle. Notice also that the sides of the triangle form circumscribed angles with the circle.



REVIEW EXAMPLES

1) $\angle PNQ$ is inscribed in circle O and $m\widehat{PQ} = 70°$.



a. What is the measure of $\angle POQ$?

b. What is the relationship between $\angle POQ$ and $\angle PNQ$?

c. What is the measure of $\angle PNQ$?

Solution:

a. The measure of a central angle is equal to the measure of the intercepted arc.

$m\angle POQ = m\widehat{PQ} = 70°$.

b. $\angle POQ$ is a central angle that intercepts $\widehat{PQ}$. $\angle PNQ$ is an inscribed angle that intercepts $\widehat{PQ}$. The measure of the central angle is equal to the measure of the intercepted arc.

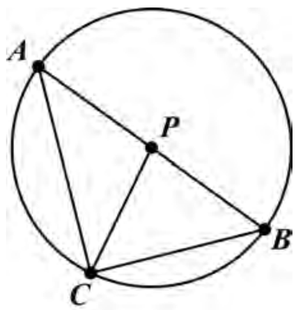
The measure of the inscribed angle is equal to one-half the measure of the intercepted arc. So $m\angle POQ = \overset{\frown}{PQ}$ and $m\angle PNQ = \frac{1}{2}\overset{\frown}{PQ}$, so $m\angle POQ = 2m\angle PNQ.$

c. From part (b), $m\angle POQ = 2m\angle PNQ$

Substitute: $70° = 2m\angle PNQ$

Divide: $35° = m\angle PNQ$

2) In circle P below, $\overline{AB}$ is a diameter.



If $m\angle APC = 100°,$ find the following:

a. $m\angle BPC$
b. $m\angle BAC$
c. $m\overset{\frown}{BC}$
d. $m\overset{\frown}{AC}$

Solution:

a. $\angle APC$ and $\angle BPC$ are supplementary, so $m\angle BPC = 180° - 100° = 80°.$

b. $\angle BAC$ is an angle in $\triangle APC.$ The sum of the measures of the angles of a triangle is 180°.
For $\triangle APC$: $m\angle APC + m\angle BAC + m\angle ACP = 180°.$
You are given that $m\angle APC = 100°.$
Substitute: $100° + m\angle BAC + m\angle ACP = 180°.$
Subtract 100° from both sides: $m\angle BAC + m\angle ACP = 80°.$
Because two sides of $\triangle APC$ are radii of the circle, $\triangle APC$ is an isosceles triangle. This means that the two base angles are congruent, so $m\angle BAC = m\angle ACP.$
Substitute $m\angle BAC$ for $m\angle ACP$: $m\angle BAC + m\angle BAC = 80°.$
Add: $2m\angle BAC = 80°$
Divide: $m\angle BAC = 40°$

You could also use the answer from part (a) to solve for $m\angle BAC$. Part (a) shows $m\angle BPC = 80°$.

Because the central angle measure is equal to the measure of the intercepted arc, $m\angle BPC = m\overset{\frown}{BC} = 80°$.

Because an inscribed angle is equal to one-half the measure of the intercepted arc, $m\angle BAC = \frac{1}{2}m\overset{\frown}{BC}$.

By substitution: $m\angle BAC = \frac{1}{2}(80°)$

Therefore, $m\angle BAC = 40°$.

c. $\angle BAC$ is an inscribed angle intercepting $\overset{\frown}{BC}$. The intercepted arc is twice the measure of the inscribed angle.

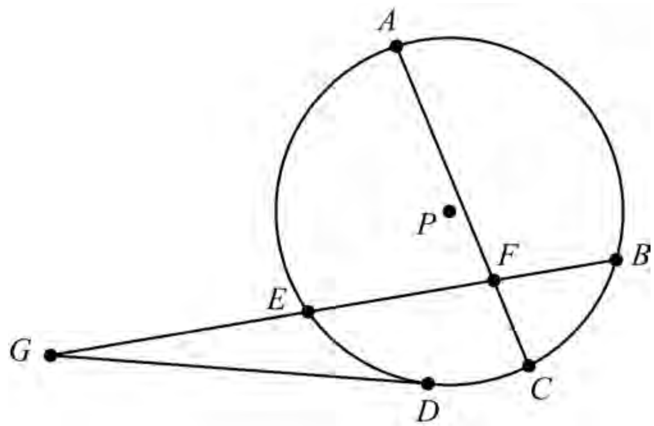
$m\overset{\frown}{BC} = 2m\angle BAC$

From part (b), $m\angle BAC = 40°$
Substitute: $m\overset{\frown}{BC} = 2 \bullet 40°$
$m\overset{\frown}{BC} = 80°$
You could also use the answer from part (a) to solve. Part (a) shows $m\angle BPC = 80°$.
Because $\angle BPC$ is a central angle that intercepts $\overset{\frown}{BC}$, $m\angle BPC = m\overset{\frown}{BC} = 80°$.

d. $\angle APC$ is a central angle intercepting $\overset{\frown}{AC}$. The measure of the intercepted arc is equal to the measure of the central angle.

$m\overset{\frown}{AC} = m\angle APC$
You are given $m\angle APC = 100°$
Substitute: $m\overset{\frown}{AC} = 100°$.

3) In circle *P* below, $\overline{DG}$ is a tangent. $AF = 8$, $EF = 6$, $BF = 4$, and $EG = 8$.



Find CF and DG.

Solution:

First, find CF. Use the fact that $\overline{CF}$ is part of a pair of intersecting chords.

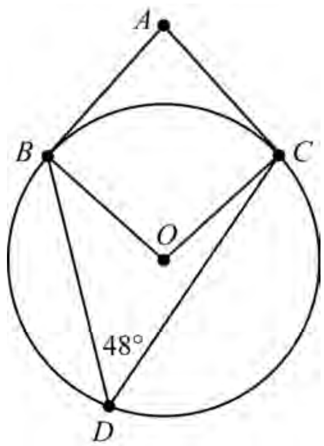
$$\begin{aligned} AF \bullet CF &= EF \bullet BF \\ 8 \bullet CF &= 6 \bullet 4 \\ 8 \bullet CF &= 24 \\ CF &= 3 \end{aligned}$$

Next, find DG. Use the fact that $\overline{DG}$ is tangent to the circle.

$$\begin{aligned} EG \bullet BG &= DG^2 \\ 8 \bullet (8+6+4) &= DG^2 \\ 8 \bullet 18 &= DG^2 \\ 144 &= DG^2 \\ \pm 12 &= DG \\ 12 &= DG \text{ (since length cannot be negative)} \end{aligned}$$

$CF = 3$ and $DG = 12$.

4) In this circle, $\overline{AB}$ is tangent to the circle at point *B*, $\overline{AC}$ is tangent to the circle at point *C*, and point *D* lies on the circle. What is $m\angle BAC$?



Solution:

Method 1

First, find the measure of angle *BOC*. Angle *BDC* is an inscribed angle, and angle *BOC* is a central angle.

$$\begin{aligned} m\angle BOC &= 2 \bullet m\angle BDC \\ &= 2 \bullet 48° \\ &= 96° \end{aligned}$$

Angle *BAC* is a circumscribed angle. Use the measure of angle *BOC* to find the measure of angle *BAC*.

$$\begin{aligned} m\angle BAC &= 180° - m\angle BOC \\ &= 180° - 96° \\ &= 84° \end{aligned}$$

Method 2

Angle *BDC* is an inscribed angle. First, find the measures of $\overset{\frown}{BC}$ and $\overset{\frown}{BDC}$.

$$\begin{aligned} m\angle BDC &= \frac{1}{2} \bullet m\overset{\frown}{BC} \\ 48° &= \frac{1}{2} \bullet m\overset{\frown}{BC} \\ 2 \bullet 48° &= m\overset{\frown}{BC} \\ 96° &= m\overset{\frown}{BC} \end{aligned}$$

$$
\begin{aligned}
m\overset{\frown}{BDC} &= 360° - m\overset{\frown}{BC} \\
&= 360° - 96° \\
&= 264°
\end{aligned}
$$

Angle *BAC* is a circumscribed angle. Use the measures of $\overset{\frown}{BC}$ and $\overset{\frown}{BDC}$ to find the measure of angle *BAC*.

$$
\begin{aligned}
m\angle BAC &= \frac{1}{2}\left(m\overset{\frown}{BDC} - m\overset{\frown}{BC}\right) \\
&= \frac{1}{2}(264° - 96°) \\
&= \frac{1}{2}(168°) \\
&= 84°
\end{aligned}
$$

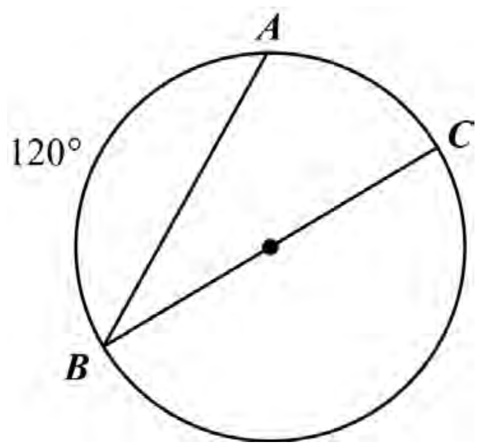
EOCT Practice Items

1) Circle P is dilated to form circle P'. Which statement is ALWAYS true?

A. The radius of circle P is equal to the radius of circle P'.
B. The length of any chord in circle P is greater than the length of any chord in circle P'.
C. The diameter of circle P is greater than the diameter of circle P'.
D. The ratio of the diameter to the circumference is the same for both circles.

[Key: D]

2) In the circle shown, $\overline{BC}$ is a diameter and $m\overset{\frown}{AB} = 120°$.



What is the measure of $\angle ABC$?

A. 15°
B. 30°
C. 60°
D. 120°

[Key: B]

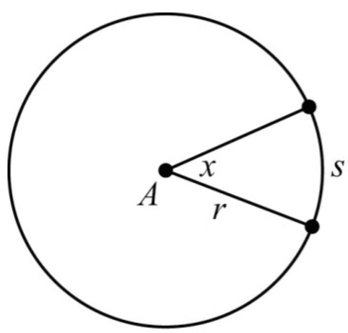
FIND ARC LENGTHS AND AREAS OF SECTORS OF CIRCLES

KEY IDEAS

1. ***Circumference*** is the distance around a circle. The formula for circumference C of a circle is $C = 2\pi r$, where r is the length of the radius of the circle.

2. ***Area*** is a measure of the amount of space a circle covers. The formula for area A of a circle is $A = \pi r^2$, where r is the length of the radius of the circle.

3. ***Arc length*** is a portion of the circumference of a circle. To find the length of an arc, divide the number of degrees in the central angle of the arc by 360°, and then multiply that amount by the circumference of the circle. The formula for arc length, s, is $s = \frac{x}{360}(2\pi r)$, where x is the degree measure of the central angle and r is the radius of the circle.



Important Tip

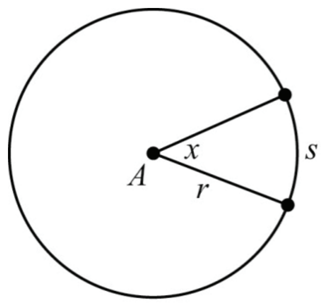
Do not confuse arc length with the measure of the arc in degrees. Arc length depends on the size of the circle because it is part of the circle. The measure of the arc is independent of (does not depend on) the size of the circle.

One way to remember the formulas for arc length is:

arc length = fraction of the circle × circumference = $\frac{x°}{360°}(2\pi r)$

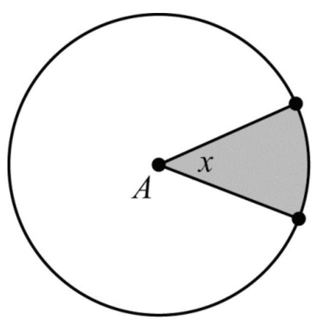
4. Angles and arcs can also be measured in ***radians***. π radians = 180°. To convert radians to degrees, multiply the radian measure by $\frac{180}{\pi}$. To convert degrees to radians, multiply the degree measure by $\frac{\pi}{180}$.

5. The length of the arc intercepted by an angle is proportional to the radius. The ratio of the arc length of a circle to the circumference of the circle is equal to the ratio of the angle measure in radians to 2π. The measure of the angle in radians is the constant of proportionality.



$$\frac{s}{2\pi r} = \frac{x}{2\pi};\ \frac{s}{r} = x;\ s = xr$$

6. A ***sector*** of a circle is the region bounded by two radii of a circle and the resulting arc between them. To find the area of a sector, divide the number of degrees in the central angle of the arc by 360°, and then multiply that amount by the area of the circle. The formula for area of a sector is $A_{sector} = \frac{x}{360}\left(\pi r^2\right)$, where x is the degree measure of the central angle and r is the radius of the circle.



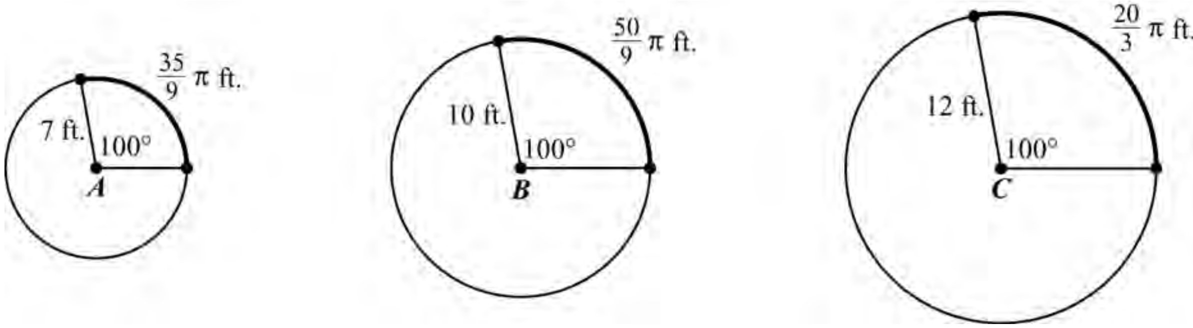
Important Tip

One way to remember the formula for area of a sector is:

area of a sector = fraction of the circle × area = $\frac{x°}{360°}\left(\pi r^2\right)$

REVIEW EXAMPLES

1) Circles A, B, and C have a central angle measuring 100°. The lengths of the radius and intercepted arc are shown.



a. What is the ratio of the radius of circle B to the radius of circle A?

b. What is the ratio of the length of the intercepted arc of circle B to the length of the intercepted arc of circle A?

c. Compare the ratios in parts (a) and (b).

d. What is the ratio of the radius of circle C to the radius of circle B?

e. What is the ratio of the length of the intercepted arc of circle C to the length of the intercepted arc of circle B?

f. Compare the ratios in parts (d) and (e).

g. Based on your observations of circles A, B, and C, what conjecture can you make about the length of the arc intercepted by a central angle and the radius?

h. What is the ratio of arc length to radius for each circle?

i. Compare the answers to part (h) with the radian measure of the central angle for each circle.

j. What conjecture can you make about the ratio of the arc length to radius?

Solution:

a. Divide the radius of circle B by the radius of circle A: $\frac{10}{7}$.

b. Divide the length of the intercepted arc of circle B by the length of the intercepted arc of circle A:

$$\frac{\frac{50}{9}\pi}{\frac{35}{9}\pi} = \frac{50}{9} \bullet \frac{9}{35} = \frac{10}{7}.$$

c. The ratios are the same.

d. Divide the radius of circle *C* by the radius of circle *B*: $\frac{12}{10} = \frac{6}{5}$.

e. Divide the length of the intercepted arc of circle *C* by the length of the intercepted arc of circle *B*: $\frac{\frac{20}{3}\pi}{\frac{50}{9}\pi} = \frac{20}{3} \bullet \frac{9}{50} = \frac{6}{5}$.

f. The ratios are the same.

g. When circles, such as circles *A*, *B*, and *C*, have the same central angle measure, the ratio of the lengths of the intercepted arcs is the same as the ratio of the radii.

h. Circle *A*: $\frac{\frac{35}{9}\pi}{7} = \frac{35}{63}\pi = \frac{5}{9}\pi$

Circle *B*: $\frac{\frac{50}{9}\pi}{10} = \frac{50}{90}\pi = \frac{5}{9}\pi$

Circle *C*: $\frac{\frac{20}{3}\pi}{12} = \frac{20}{36}\pi = \frac{5}{9}\pi$

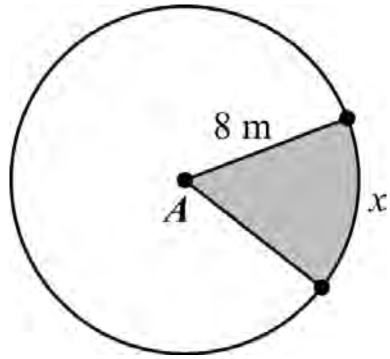
i. Use the conversion factor $\frac{\pi}{180}$ to find the radian measure of the central angle.

$$100°\left(\frac{\pi}{180}\right) = \frac{100}{180}\pi = \frac{5}{9}\pi$$

The radian measure of the central angle is the same as the ratios of the arc lengths to radii.

j. When circles have the same central angle measure, the ratios of the lengths of the intercepted arcs and the radii are proportional and the constant of proportionality is the radian measure of the central angle.

2) Circle A is shown.



If $x = 50°$, what is the area of the shaded sector of circle A?

Solution:

To find the area of the sector, divide the measure of the central angle of the arc by 360°, and then multiply that amount by the area of the circle. The arc measure, x, is equal to the measure of the central angle. The formula for the area of a circle is $A = \pi r^2$.

$A_{sector} = \frac{x}{360}\left(\pi r^2\right)$ Area of sector of a circle with radius r and central angle x in degrees.

$A_{sector} = \frac{50}{360}\left(\pi(8)^2\right)$ Substitute 50° for x and 8 for r.

$A_{sector} = \frac{5}{36}\pi(64)$ Rewrite the fraction and the power.

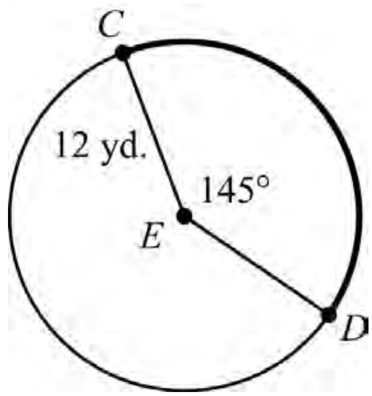
$A_{sector} = \frac{320}{36}\pi$ Multiply.

$A_{sector} = \frac{80}{9}\pi$ Rewrite.

The area of the sector is $\frac{80}{9}\pi$ square meters.

EOCT Practice Items

1) Circle *E* is shown.



What is the length of $\overset{\frown}{CD}$?

A. $\frac{29}{72}\pi$ yd.

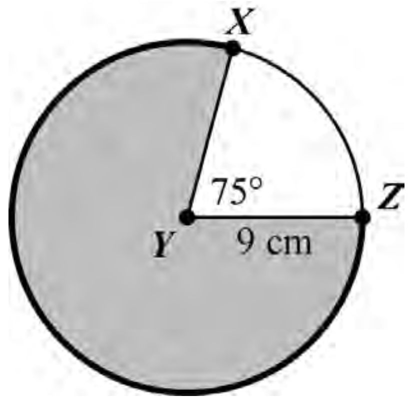
B. $\frac{29}{6}\pi$ yd.

C. $\frac{29}{3}\pi$ yd.

D. $\frac{29}{2}\pi$ yd.

[Key: C]

2) Circle *Y* is shown.



What is the area of the shaded part of the circle?

A. $\frac{57}{4}\pi\ \text{cm}^2$

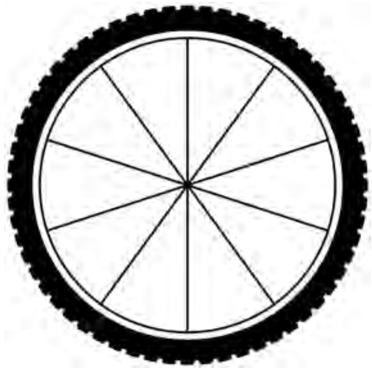
B. $\frac{135}{8}\pi\ \text{cm}^2$

C. $\frac{405}{8}\pi\ \text{cm}^2$

D. $\frac{513}{8}\pi\ \text{cm}^2$

[Key: D]

3) The spokes of a bicycle wheel form 10 congruent central angles. The diameter of the circle formed by the outer edge of the wheel is 18 inches.



What is the length, to the nearest 0.1 inch, of the outer edge of the wheel between two consecutive spokes?

A. 1.8 inches
B. 5.7 inches
C. 11.3 inches
D. 25.4 inches

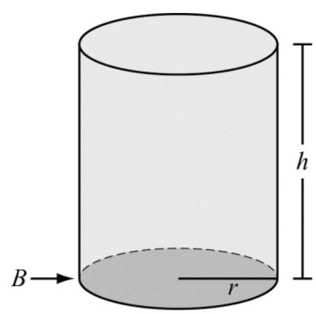
[Key: B]

EXPLAIN VOLUME FORMULAS AND USE THEM TO SOLVE PROBLEMS

KEY IDEAS

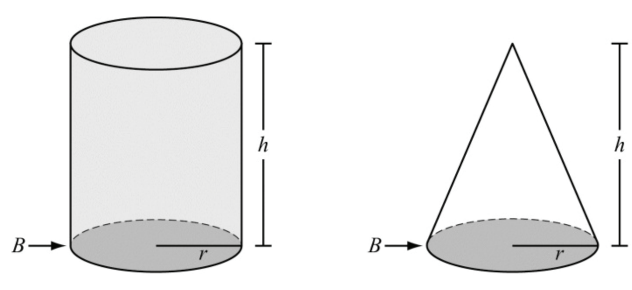
1. The ***volume*** of a figure is a measure of how much space it takes up. Volume is a measure of capacity.

2. The formula for the volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height. The volume formula can also be given as $V = Bh$, where B is the area of the base. In a cylinder, the base is a circle and the area of a circle is given by $A = \pi r^2$. Therefore, $V = Bh = \pi r^2 h$.

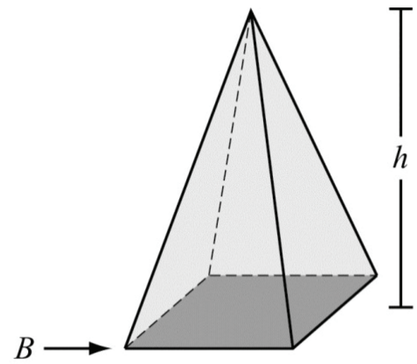


3. When a cylinder and a cone have congruent bases and equal heights, the volume of exactly three cones will fit into the cylinder. So, for a cone and cylinder that have the same radius r and height h, the volume of the cone is one-third of the volume of the cylinder.

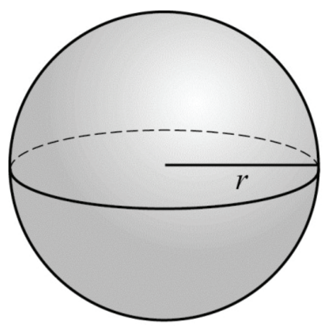
   The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$, where r is the radius and h is the height.



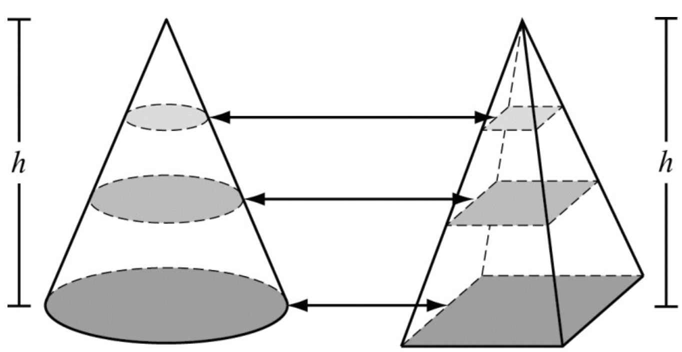
4. The formula for the volume of a pyramid is $V = \frac{1}{3}Bh$, where B is the area of the base and h is the height.



5. The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$, where r is the radius.

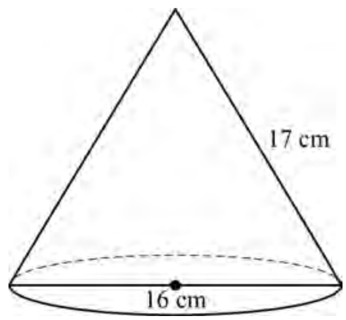


6. Cavalieri's principle states that if two solids are between parallel planes and all cross sections at equal distances from their bases have equal areas, the solids have equal volumes. For example, this cone and this pyramid have the same height and the cross sections have the same area, so they have equal volumes.



REVIEW EXAMPLES

1) What is the volume of the cone shown below?



Solution:

The diameter of the cone is 16 cm. So the radius is 16 cm ÷ 2 = 8 cm. Use the Pythagorean Theorem, $a^2 + b^2 = c^2$, to find the height of the cone. Substitute 8 for b and 17 for c and solve for a:

$$a^2 + 8^2 = 17^2$$
$$a^2 + 64 = 289$$
$$a^2 = 225$$
$$a = 15$$

The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. Substitute 8 for r and 15 for h:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(8)^2(15).$$

The volume is 320π cm^3.

2) A sphere has a radius of 3 feet. What is the volume of the sphere?

Solution:

The formula for the volume of a sphere is $V = \frac{4}{3}\pi r^3$. Substitute 3 for r and solve.

$$V = \frac{4}{3}\pi r^3$$
$$V = \frac{4}{3}\pi(3)^3$$
$$V = \frac{4}{3}\pi(27)$$
$$V = 36\pi \text{ ft}^3$$

3) A cylinder has a radius of 10 cm and a height of 9 cm. A cone has a radius of 10 cm and a height of 9 cm. Show that the volume of the cylinder is three times the volume of the cone.

Solution:

The formula for the volume of a cylinder is $V = \pi r^2 h$. Substitute 10 for r and 9 for h:

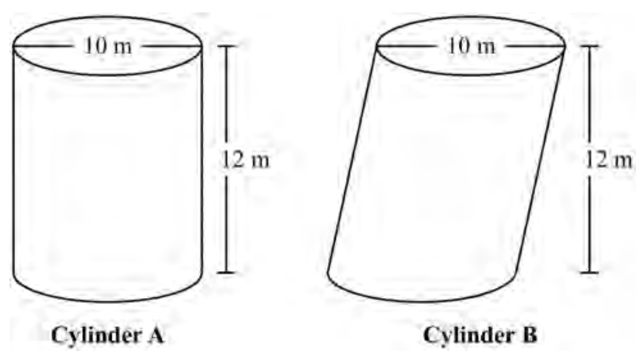
$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(10)^2(9) \\ &= \pi(100)(9) \\ &= 900\pi \text{ cm}^3 \end{aligned}$$

The formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h.$ Substitute 10 for r and 9 for h:

$$\begin{aligned} V &= \frac{1}{3}\pi r^2 h \\ &= \frac{1}{3}\pi(10)^2(9) \\ &= \frac{1}{3}\pi(100)(9) \\ &= 300\pi \text{ cm}^3 \end{aligned}$$

Divide: $900\pi \div 300\pi = 3$.

4) Cylinder A and cylinder B are shown below. What is the volume of each cylinder?



Solution:

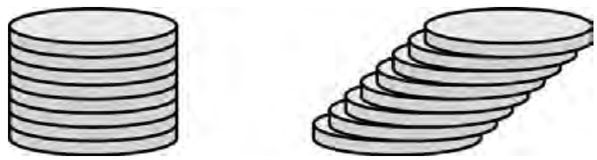
To find the volume of Cylinder A, use the formula for the volume of a cylinder, which is $V = \pi r^2 h$. Divide the diameter by 2 to find the radius: $10 \div 2 = 5$. Substitute 5 for r and 12 for h:

$$\begin{aligned} V_{\text{Cylinder A}} &= \pi r^2 h \\ &= \pi(5)^2(12) \\ &= \pi(25)(12) \\ &= 300\pi \text{ m}^3 \\ &\approx 942 \text{ m}^3 \end{aligned}$$

Notice that Cylinder B has the same height and the same radius as Cylinder A. The only difference is that Cylinder B is slanted. For both cylinders, the cross section at every plane parallel to the bases is a circle with the same area. By Cavalieri's principle, the cylinders have the same volume; therefore, the volume of Cylinder B is 300π m^3, or about 942 m^3.

EOCT Practice Items

1) Jason constructed two cylinders using solid metal washers. The cylinders have the same height, but one of the cylinders is slanted as shown.



Which statement is true about Jason's cylinders?

A. The cylinders have different volumes because they have different radii.
B. The cylinders have different volumes because they have different surface areas.
C. The cylinders have the same volume because each of the washers has the same height.
D. The cylinders have the same volume because they have the same cross-sectional area at every plane parallel to the bases.

[Key: D]

2) What is the volume of a cylinder with a radius of 3 in. and a height of $\frac{9}{2}$ in.?

A. $\frac{81}{2}\pi$ in.^{3}

B. $\frac{27}{4}\pi$ in.^{3}

C. $\frac{27}{8}\pi$ in.^{3}

D. $\frac{9}{4}\pi$ in.^{3}

[Key: A]

Unit 4: Extending the Number System

This unit investigates properties of rational exponents and how the concept of rational exponents relates to radicals. Closure properties are explored in terms of number systems as well as polynomials. In this unit, the set of complex numbers is introduced. Operations with complex numbers are performed.

KEY STANDARDS

Extend the properties of exponents to rational exponents

MCC9-12.N.RN.1 Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{(1/3)}$ *to be the cube root of 5 because we want* $[5^{(1/3)}]^3 = 5^{[(1/3) \times 3]}$ *to hold, so* $[5^{(1/3)}]^3$ *must equal 5.*

MCC9-12.N.RN.2 Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Use properties of rational and irrational numbers

MCC9-12.N.RN.3 Explain why the sum or product of rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

Perform arithmetic operations with complex numbers

MCC9-12.N.CN.1 Know there is a complex number i such that $i^2 = -1$, and every complex number has the form $a + bi$ with a and b real.

MCC9-12.N.CN.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

Perform arithmetic operations on polynomials

MCC9-12.A.APR.1 Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials. (*Focus on polynomial expressions that simplify to forms that are linear or quadratic in a positive integer power of x.*)

EXTEND THE PROPERTIES OF EXPONENTS TO RATIONAL EXPONENTS

KEY IDEAS

1. The ***nth root*** of a number is the number that must be multiplied by itself *n* times to equal a given value. It can be notated with radicals and indices or with rational exponents. When a root does not have an index, the index is assumed to be 2.

$$\sqrt[\text{index}]{\text{radicand}}$$

Examples:

$\sqrt{49}\sqrt{7 \bullet 7} = 7$

$\sqrt[3]{64}\sqrt{4 \bullet 4 \bullet 4} = 4$

$\sqrt{18} = \sqrt{9 \bullet 2} = 3\sqrt{2}$

$\sqrt{x^5} = \sqrt{x^4 \bullet x} = x^2\sqrt{x}$

2. ***Rational exponents*** are a way to express roots as powers. The denominator of a rational exponent is the index of the radical, and the numerator is the exponent of the radicand.

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}$$

Examples:

$\sqrt{x} = x^{\frac{1}{2}}$

$\sqrt[3]{a^5} = a^{\frac{5}{3}}$

3. To rewrite radicals, you can use rational exponents and solve the problem using the laws of exponents, where $x \neq 0$ and a and b are real numbers.

- Product of powers rule: $x^a \bullet x^b = x^{a+b}$
- Quotient of powers rule: $\frac{x^a}{x^b} = x^{a-b}$
- Power of powers rule: $\left(x^a\right)^b = x^{a \bullet b}$
- $x^0 = 1,\ x > 0$
- $x^{-1} = \frac{1}{x}$

Examples:

$$x^3 \bullet x^5 = x^{3+5} = x^8$$

$$\frac{x^5}{x^3} = x^{5-3} = x^2$$

$$\left(x^3\right)^5 = x^{3 \bullet 5} = x^{15}$$

$$5^0 = 1$$

$$5^{-1} = \frac{1}{5}$$

4. To rewrite square root expressions, you can use properties of square roots where *a* and *b* are real numbers with *a* > 0 and *b* > 0.

- Product property: $\sqrt{ab} = \sqrt{a} \bullet \sqrt{b}$
- Quotient property: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Examples:

$$\sqrt{32} = \sqrt{16 \bullet 2} = \sqrt{16} \bullet \sqrt{2} = 4\sqrt{2}$$

$$\sqrt{\frac{9}{25}} = \frac{\sqrt{9}}{\sqrt{25}} = \frac{3}{5}$$

5. One common way to rewrite square root expressions is so that no radical has a perfect square factor and there is no radical in a denominator. A square root in a denominator can be eliminated by multiplying the numerator and denominator of the fraction by the square root that appears in the denominator.

Example:

Rewrite $\frac{3}{\sqrt{5}}$:

$$\frac{3}{\sqrt{5}}\left(\frac{\sqrt{5}}{\sqrt{5}}\right) = \frac{3\sqrt{5}}{\sqrt{25}} = \frac{3\sqrt{5}}{5}$$

6. Two radical expressions that have the same index and the same radicand are called ***like radicals.*** To add or subtract like radicals, you can use the Distributive Property.

Example:

$\sqrt{8}+\sqrt{2}$	
$\sqrt{4\bullet 2}+\sqrt{2}$	Factor out the perfect square.
$\sqrt{4}\bullet\sqrt{2}+\sqrt{2}$	Use the product property of square roots.
$2\sqrt{2}+\sqrt{2}$	Compute the square root.
$(2+1)\sqrt{2}$	Distributive Property.
$3\sqrt{2}$	Add.

Important Tip

Remember, when using rational exponents, to find a common denominator before adding or subtracting the exponents.

REVIEW EXAMPLES

1) Rewrite $\dfrac{\sqrt[4]{x}\bullet\sqrt[6]{x}}{\sqrt[3]{x}}$, where $x>0$, using rational exponents.

Solution:

Rewrite the expression using rational exponents: $\dfrac{x^{\frac{1}{4}}\bullet x^{\frac{1}{6}}}{x^{\frac{1}{3}}}$.

Use the product of powers rule to multiply the expressions in the numerator:

$$\frac{x^{\frac{1}{4}+\frac{1}{6}}}{x^{\frac{1}{3}}}=\frac{x^{\frac{3}{12}+\frac{2}{12}}}{x^{\frac{1}{3}}}=\frac{x^{\frac{5}{12}}}{x^{\frac{1}{3}}}.$$

Use the quotient of powers rule to divide: $\dfrac{x^{\frac{5}{12}}}{x^{\frac{1}{3}}}=x^{\frac{5}{12}-\frac{1}{3}}=x^{\frac{5}{12}-\frac{4}{12}}=x^{\frac{1}{12}}$.

2) Rewrite $\sqrt[5]{243x^{25}y^{12}z^{-8}}$ using positive rational exponents.

Solution:

$$\sqrt[5]{243x^{25}y^{12}z^{-8}}=243^{\frac{1}{5}}x^{\frac{25}{5}}y^{\frac{12}{5}}z^{\frac{-8}{5}}=\frac{3x^5y^{\frac{12}{5}}}{z^{\frac{8}{5}}}.$$

3) Look at the formula below.

$$p^2 = d^3$$

a. Solve the formula for *d*.

b. Write the formula using rational exponents.

Solution:

a. Take the cube root of both sides: $p^2 = d^3$

$$\sqrt[3]{p^2} = \sqrt[3]{d^3}$$
$$\sqrt[3]{p^2} = d$$

b. Write the formula for *d* using a rational exponent.

$$p^{\frac{2}{3}} = d$$

4) Rewrite $\sqrt{2}\left(\sqrt{12}-\sqrt{3}\right)$.

Solution:

$\sqrt{2}\left(\sqrt{12}-\sqrt{3}\right)$	Original expression.
$\sqrt{2} \bullet \sqrt{12} - \sqrt{2} \bullet \sqrt{3}$	Distributive Property.
$\sqrt{2} \bullet \sqrt{4 \bullet 3} - \sqrt{2} \bullet \sqrt{3}$	Factor out the perfect square.
$\sqrt{2} \bullet \sqrt{4} \bullet \sqrt{3} - \sqrt{2} \bullet \sqrt{3}$	Product Property.
$2 \bullet \sqrt{2} \bullet \sqrt{3} - \sqrt{2} \bullet \sqrt{3}$	Evaluate the square root.
$2\sqrt{6} - \sqrt{6}$	Use the product property of square roots.
$\sqrt{6}$	Subtract.

5) Write $\sqrt{\frac{25}{18}}$ in an equivalent form where no radical has a perfect square factor, and there is no radical in the denominator.

Solution:

$\sqrt{\frac{25}{18}}$ Original expression.

$\frac{\sqrt{25}}{\sqrt{18}}$ Quotient property.

$\frac{5}{\sqrt{18}}$ Evaluate the square root.

$\frac{5}{\sqrt{9 \bullet 2}}$ Factor out the perfect square.

$\frac{5}{\sqrt{9} \bullet \sqrt{2}}$ Product Property.

$\frac{5}{3\sqrt{2}}$ Evaluate the square root.

To eliminate the radical in the denominator, multiply the numerator and denominator by $\sqrt{2}$.

$\frac{5}{3\sqrt{2}} \bullet \frac{\sqrt{2}}{\sqrt{2}}$ Multiply.

$\frac{5\sqrt{2}}{3\sqrt{4}}$ Product Property.

$\frac{5\sqrt{2}}{3(2)}$ Evaluate the square root.

$\frac{5\sqrt{2}}{6}$ Multiply.

6) Write $\sqrt{\left(4p^2\right)^3}$ in an equivalent form without a square root. Assume that *p* is non-negative.

Solution:

$\sqrt{\left(4p^2\right)^3}$ Original expression.

$\sqrt{64p^6}$ Power of powers rule.

$\left(64p^6\right)^{\frac{1}{2}}$ Rewrite the square root using a fractional exponent.

$8p^3$ Evaluate.

EOCT Practice Items

1) Which shows $\sqrt[4]{x^4}+\sqrt[4]{16x^2}+\sqrt[4]{x}$ using rational exponents for all positive values of *x*?

A. $x^{16}+16x^8+x^4$

B. $x+2x^{\frac{1}{2}}+x^{\frac{1}{4}}$

C. $x+4x^{\frac{1}{2}}+x^{\frac{1}{4}}$

D. $x^{\frac{1}{16}}+2x^{\frac{1}{8}}+x^{\frac{1}{4}}$

[Key: B]

2) Which expression is equivalent to $\frac{\sqrt{x}}{x^3}$?

A. $x^{\frac{5}{2}}$

B. $\sqrt{x^5}$

C. $\frac{1}{\sqrt{x^5}}$

D. $\frac{1}{x\sqrt{x}}$

[Key: C]

3) Which expression is equivalent to $\sqrt[3]{64x^{\frac{6}{7}}}$?

A. $4x^{\frac{2}{7}}$

B. $4x^{\frac{18}{7}}$

C. $64x^{\frac{2}{7}}$

D. $64x^{18}$

[Key: A]

4) Which expression is equivalent to $\sqrt{32} - \sqrt{8}$?

A. $2\sqrt{2}$
B. $6\sqrt{2}$
C. $2\sqrt{6}$
D. $2\sqrt{10}$

[Key: A]

5) Which expression is equivalent to $\sqrt{\dfrac{16}{27}}$?

A. $\dfrac{4\sqrt{3}}{3}$
B. $\dfrac{2\sqrt{3}}{3}$
C. $\dfrac{3\sqrt{3}}{4}$
D. $\dfrac{4\sqrt{3}}{9}$

[Key: D]

USE PROPERTIES OF RATIONAL AND IRRATIONAL NUMBERS

KEY IDEAS

1. A ***rational number*** is a real number that can be represented as a ratio $\frac{p}{q}$, such that p and q are both integers and $q \neq 0$. All rational numbers can be expressed as a terminating or repeating decimal.

Examples:

-0.5, 0, 7, $\frac{3}{2}$, $0.2\overline{6}$

2. The sum, product, or difference of two rational numbers is always a rational number. The quotient of two rational numbers is always rational when the divisor is not zero.

Example:

Show that the sum of two rational numbers is rational.

Solution:

Let a and b be rational numbers. Try to show that $a + b$ is rational.

Let $a = \frac{p}{q}$, where p and q are integers and $q \neq 0$.

Let $b = \frac{m}{n}$, where m and n are integers and $n \neq 0$.

Substitute $\frac{p}{q}$ and $\frac{m}{n}$ for a and b. To add, find a common denominator.

$$a + b = \frac{p}{q} + \frac{m}{n} = \frac{np}{nq} + \frac{mq}{nq}$$

$$= \frac{np + mq}{nq}$$

The set of integers is closed under multiplication, so the products np, mq, and nq are all integers. The set of integers is also closed under addition, so the sum $np + mq$ is also an integer. This means that $\frac{np+mq}{nq}$ is an integer divided by an integer and, by definition, is rational. So, the sum of a and b is rational.

3. An ***irrational number*** is a real number that cannot be expressed as a ratio $\frac{p}{q}$, such that p and q are both integers and $q \neq 0$. Irrational numbers cannot be represented by terminating or repeating decimals.

Examples:

$\sqrt{3},\ \pi,\ \frac{\sqrt{5}}{2}$

4. The sum of an irrational number and a rational number is always irrational. The product of a nonzero rational number and an irrational number is always irrational.

Example:

Show that the sum of an irrational number and a rational number is irrational.

Solution:

Let a be an irrational number, and let b be a rational number. Suppose that the sum of a and b is a rational number, c. If you can show that this is not true, it is the same as proving the original statement.

Let $b = \frac{p}{q}$, where p and q are integers and $q \neq 0$.

Let $c = \frac{m}{n}$ where m and n are integers and $n \neq 0$.

Substitute $\frac{p}{q}$ and $\frac{m}{n}$ for b and c. Then subtract to find a.

$$a + b = c$$

$$a + \frac{p}{q} = \frac{m}{n}$$

$$a = \frac{m}{n} - \frac{p}{q}$$

$$a = \frac{mq}{nq} - \frac{pn}{nq}$$

$$a = \frac{mq - pn}{nq}$$

The set of integers is closed under multiplication and subtraction, so $\frac{mq - pn}{nq}$ is an integer divided by an integer. This means that *a* is rational. However, *a* was assumed to be irrational, so this is a contradiction. This means that *c* must be irrational. So, the sum of an irrational number and a rational number is irrational.

REVIEW EXAMPLES

1) Look at the expression below.

$$(5 + \sqrt{2}) + 2\sqrt{2}$$

Is the value of the expression rational or irrational? Explain.

Solution:

Use the associative property to add like terms.

$$\begin{aligned}(5 + \sqrt{2}) + 2\sqrt{2} &= 5 + \left(\sqrt{2} + 2\sqrt{2}\right) \\ &= 5 + 3\sqrt{2}\end{aligned}$$

$3\sqrt{2}$ is the product of a rational number, 3, and an irrational number, $\sqrt{2}$. The product of an irrational number and a rational number is always irrational. So, $3\sqrt{2}$ is irrational. 5 is a rational number and $3\sqrt{2}$ is an irrational number. The sum of a rational number and an irrational number is always irrational, so the value of the expression $5 + 3\sqrt{2}$ is irrational.

2) Explain why the product π • 5 is irrational.

Solution:

The product π • 5 = 5π. Assume 5π is a rational number. The quotient of 2 rational numbers is rational. Therefore $\frac{5\pi}{5}$ would be a rational number, and $\frac{5\pi}{5} = \pi.$ This would mean that π is a rational number. This is a contradiction because π is an irrational number. So, 5π is irrational.

3) Is the value of the expression $\sqrt{8}(5\sqrt{8} + \sqrt{2})$ rational or irrational? Explain how you found your answer.

Solution:

$$\begin{aligned}\sqrt{8}\left(5\sqrt{8} + \sqrt{2}\right) &= 5\sqrt{8} \bullet \sqrt{8} + \sqrt{2} \bullet \sqrt{8} \\ &= 5\sqrt{64} + \sqrt{16} \\ &= 5 \bullet 8 + 4 \\ &= 40 + 4 \\ &= 44\end{aligned}$$

The value of the expression is 44, which is rational.

EOCT Practice Items

1) Which expression has a value that is a rational number?

A. $\sqrt{10} + 16$

B. $2\left(\sqrt{5} + \sqrt{7}\right)$

C. $\sqrt{9} + \sqrt{4}$

D. $\sqrt{3} + 0$

[Key: C]

2) Which statement is true about the value of $\left(\sqrt{8} + 4\right) \bullet 4$?

A. It is rational, because the product of two rational numbers is rational.

B. It is rational, because the product of a rational number and an irrational number is rational.

C. It is irrational, because the product of two irrational numbers is irrational.

D. It is irrational, because the product of an irrational number and a rational number is irrational.

[Key: D]

3) Let *a* be a nonzero rational number and *b* be an irrational number. Which of these MUST be a rational number?

A. $b + 0$

B. $a + a$

C. $a + b$

D. $b + b$

[Key: B]

PERFORM ARITHMETIC OPERATIONS ON POLYNOMIALS

KEY IDEAS

1. A ***polynomial*** is an expression made from one or more terms that involve constants, variables, and exponents.

Examples:

$3x$; $x^3 + 5x^2 + 4$; $a^2b - 2ab + b^2$

2. To add and subtract polynomials, combine like terms. In a polynomial, like terms have the same variables and are raised to the same powers.

Examples:

$$\begin{aligned} 7x + 6 + 5x - 3 &= 7x + 5x + 6 - 3 \\ &= 12x + 3 \end{aligned}$$

$$\begin{aligned} 13a + 1 - (5a - 4) &= 13a + 1 - 5a + 4 \\ &= 13a - 5a + 1 + 4 \\ &= 8a + 5 \end{aligned}$$

3. To multiply polynomials, use the Distributive Property. Multiply every term in the first polynomial by every term in the second polynomial.

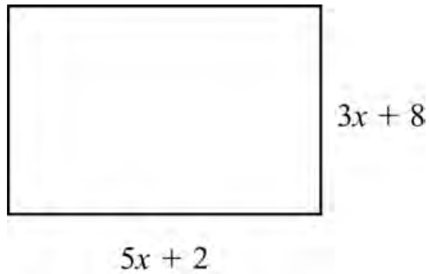
Example:

$$\begin{aligned} (x+5)(x-3) &= (x)(x) + (-3)(x) + (5)(x) + (5)(-3) \\ &= x^2 - 3x + 5x - 15 \\ &= x^2 + 2x - 15 \end{aligned}$$

4. Polynomials are closed under addition, subtraction, and multiplication, similar to the set of integers. This means that the sum, difference, or product of two polynomials is always a polynomial.

REVIEW EXAMPLES

1) The dimensions of a rectangle are shown.



What is the perimeter of the rectangle?

Solution:

Substitute $5x + 2$ for l and $3x + 8$ for w into the formula for the perimeter of a rectangle:

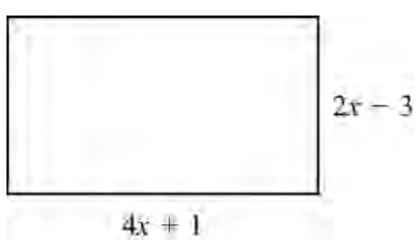
$P = 2l + 2w$
$P = 2(5x + 2) + 2(3x + 8)$
$P = 10x + 4 + 6x + 16$
$P = 10x + 6x + 4 + 16$
$P = 16x + 20$

2) Rewrite the expression $(x^3 + 2x^2 - x) - (-x^3 + 2x^2 + 6)$.

Solution:

Combine like terms:
$(x^3 + 2x^2 - x) - (-x^3 + 2x^2 + 6) = (x^3 - (-x^3)) + (2x^2 - 2x^2) + (-x) + (-6) = 2x^3 - x - 6.$

3) The dimensions of a patio, in feet, are shown below.



What is the area of the patio, in square feet?

Solution:

Substitute $4x + 1$ for l and $2x - 3$ for w into the formula for the area of a rectangle:

$A = lw$

$A = (4x + 1)(2x - 3)$

$A = 8x^2 - 12x + 2x - 3$

$A = 8x^2 - 10x - 3$ square feet

EOCT Practice Items

1) What is the product of $7x - 4$ and $8x + 5$?

A. $15x + 1$

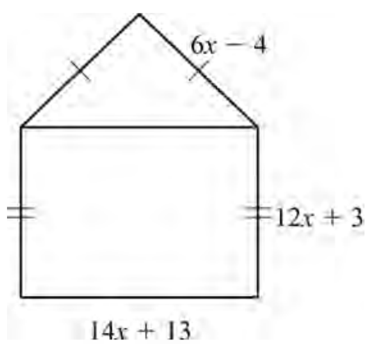
B. $30x + 2$

C. $56x^2 + 3x - 20$

D. $56x^2 - 3x + 20$

[Key: C]

2) A model of a house is shown.



What is the perimeter, in units, of the model?

A. $32x + 12$

B. $46x + 25$

C. $50x + 11$

D. $64x + 24$

[Key: C]

3) Which has the same value as the expression $(8x^2 + 2x - 6) - (5x^2 - 3x + 2)$?

A. $3x^2 - x - 4$

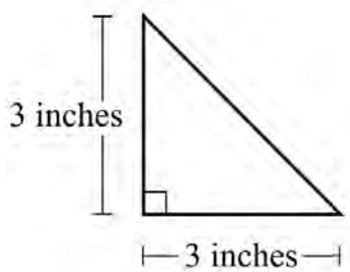
B. $3x^2 + 5x - 8$

C. $13x^2 - x - 8$

D. $13x^2 - 5x - 4$

[Key: B]

4) Kelly makes two different-sized ceramic tiles in the shape of right isosceles triangles. This diagram shows the leg lengths of the small tile.



Kelly makes a larger tile by increasing the length of each leg of the small tile by x inches. Which expression represents the length, in inches, of the hypotenuse of the large tile?

A. $18 + x$

B. $(x + 3)^2$

C. $(x + 3)\sqrt{2}$

D. $3\sqrt{2} + x$

[Key: C]

PERFORM ARITHMETIC OPERATIONS WITH COMPLEX NUMBERS

KEY IDEAS

1. An ***imaginary number*** is a number whose square is less than zero. An imaginary number can be written as a real number multiplied by the imaginary unit, *i*, where $i = \sqrt{-1}$ and $i^2 = -1$.

   Examples:

   $\sqrt{-25} = \sqrt{25} \bullet \sqrt{-1} = 5i$

   $\sqrt{-48} = \sqrt{48} \bullet \sqrt{-1} = \left(4\sqrt{3}\right)\sqrt{-1} = 4i\sqrt{3}$

2. The powers of *i* form a repeating pattern as shown.

$$
\begin{aligned}
i^0 &= 1 \\
i^1 &= i \\
i^2 &= -1 \\
i^3 &= i^2 \bullet i = -1 \bullet i = -i \\
i^4 &= i^2 \bullet i^2 = -1 \bullet -1 = 1 \\
i^5 &= i^4 \bullet i = 1 \bullet i = i \\
i^6 &= i^4 \bullet i^2 = 1 \bullet -1 = -1 \\
&\vdots \qquad \vdots
\end{aligned}
$$

3. A ***complex number*** is the sum of a real number and an imaginary number, in the form *a* + *bi*, where *a* and *b* are real numbers and *i* is the imaginary unit.

4. To add (or subtract) complex numbers, add (or subtract) the real parts and add (or subtract) the imaginary parts.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

   This is similar to combining like terms when adding or subtracting polynomials.

   Example:

   $(2 + 3i) + (4 + 5i) = (2 + 4) + (3i + 5i) = (2 + 4) + (3 + 5)i = 6 + 8i$

5. To multiply complex numbers, use the Distributive Property. Multiply each term of the first complex number by each term in the second complex number.

$$\begin{aligned}(a+bi)(c+di) &= ac+adi+bci+bdi^2\\ &= ac+adi+bci+bd(-1)\\ &= ac+adi+bci-bd\\ &= (ac-bd)+(adi+bci)\\ &= (ac-bd)+(ad+bc)i\end{aligned}$$

REVIEW EXAMPLES

1) Subtract: $(5 + 7i) – (8 – 4i)$. Identify the real part and imaginary part of the difference.

Solution:

First rewrite the expression:

$(5+7i)-(8-4i) = 5+7i-8+4i$	Distributive Property.
$= 5-8+7i+4i$	Commutative Property.
$= -3+11i$	

For complex number $a + bi$, the real part is a and the imaginary part is b. For $–3 + 11i$, the real part is –3 and the imaginary part is 11.

2) Rewrite the expression $i^2(3i – 7)$ in the form $a + bi$, and justify each step.

Solution:

Use properties and math operations to rewrite the expression.

$i^2(3i – 7) = –1(3i – 7)$	Substitute –1 for i^2.
$= –3i + 7$	Distributive Property.
$= 7 – 3i$	Commutative Property of Addition.

3) Rewrite the expression $(11i^4 + 2i^3) - (2i^4 - 6i^3)$ in the form $a + bi$, and justify each step.

Solution:

Use properties and operations to rewrite the expression.

$(11i^4 + 2i^3) - (2i^4 - 6i^3) = 11i^4 + 2i^3 + (-1)(2i^4) + (-1)(-6i^3)$	Distributive Property.
$= 11i^4 + 2i^3 - 2i^4 + 6i^3$	Multiply.
$= 11i^4 - 2i^3 + 2i^4 + 6i^3$	Commutative Property of Addition.
$= 9i^4 + 8i^3$	Combine like terms.
$= 9(1) + 8(-i)$	Substitute 1 for i^4 and $-i$ for i^3.
$= 9 - 8i$	Rewrite in $a + bi$ form.

4) Multiply $(6 + 4i)(8 - 3i)$.

Solution:

Use the Distributive Property to find the product.

$(6+4i)(8-3i) = 6 \bullet 8 + 6 \bullet (-3i) + 4i \bullet 8 + 4i \bullet (-3i)$	Distributive Property.
$(6+4i)(8-3i) = 6 \bullet 8 - 6 \bullet 3i + 4 \bullet 8i - 4 \bullet 3i^2$	Commutative Property.
$= 48 - 18i + 32i - 12(-1)$	Multiply.
$= 48 - 18i + 32i + 12$	Multiply.
$= 48 + 12 - 18i + 32i$	Commutative Property of Addition.
$= (48 + 12) + (-18 + 32)i$	Associative Property of Addition and Distributive Property.
$= 60 + 14i$	Add.

EOCT Practice Items

1) Which has the same value as $-i^5 + i^3$?

A. $-2i$
B. -2
C. 2
D. $2i$

[Key: A]

2) Let $r = 4 + i$ and $s = 1 - i$. What is the value of $r^2 - s$?

A. $14 + i$
B. $15 + i$
C. $14 + 7i$
D. $14 + 9i$

[Key: D]

3) Which has the same value as $(5 - 3i)(-4 + 2i)$?

A. $-26 - 2i$
B. $-26 + 22i$
C. $-14 - 2i$
D. $-14 + 22i$

[Key: D]

Unit 5: Quadratic Functions

This unit investigates quadratic functions. Students study the structure of expressions and write expressions in equivalent forms. They solve quadratic equations by inspection, by completing the square, by factoring, and by using the Quadratic Formula. Some quadratic equations will have complex solutions. Students also graph quadratic functions and analyze characteristics of those functions, including end behavior. They write functions for various situations and build functions from other functions, using operations as needed. Given bivariate data, students fit a function to the data and use it to make predictions.

KEY STANDARDS

Use complex numbers in polynomial identities and equations

MCC9-12.N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.

Interpret the structure of expressions

MCC9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.SSE.2 Use the structure of an expression to identify ways to rewrite it. For example, see $x^4 – y^4$ as $(x^2)^2 – (y^2)^2$, thus recognizing it as a difference of squares that can be factored as $(x^2 – y^2)(x^2 + y^2)$. (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Write expressions in equivalent forms to solve problems

MCC9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.SSE.3a Factor a quadratic expression to reveal the zeros of the function it defines.★

MCC9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.★

Create equations that describe numbers or relationships

MCC9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from ~~linear and~~ quadratic functions, ~~and simple rational and exponential functions~~.★

MCC9-12.A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. *For example, rearrange Ohm's law V = IR to highlight resistance R.*★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Solve equations and inequalities in one variable

MCC9-12.A.REI.4 Solve quadratic functions in one variable.

MCC9-12.A.REI.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

MCC9-12.A.REI.4b Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations

MCC9-12.A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line* $y = –3x$ *and the circle* $x^2 + y^2 = 3$.

Interpret functions that arise in applications in terms of the context

MCC9-12.F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; ~~and periodicity~~.★

MCC9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.*★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Analyze functions using different representations

MCC9-12.F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.IF.7a Graph ~~linear and~~ quadratic functions and show intercepts, maxima, and minima.★

MCC9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

MCC9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.* (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Build a function that models a relationship between two quantities

MCC9-12.F.BF.1 Write a function that describes a relationship between two quantities.★ (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.BF.1a Determine an explicit expression, a recursive process, or steps for calculation from a context. (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

MCC9-12.F.BF.1b Combine standard function types using arithmetic operations. *For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model.* (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Build new functions from existing functions

MCC9-12.F.BF.3 Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

Include recognizing even and odd functions from their graphs and algebraic expressions for them. (*Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra*.)

Construct and compare linear, quadratic, and exponential models to solve problems

MCC9-12.F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Summarize, represent, and interpret data on two categorical and quantitative variables

MCC9-12.S.ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★

MCC9-12.S.ID.6a Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize ~~linear~~, quadratic, ~~and exponential~~ models.★

USE COMPLEX NUMBERS IN POLYNOMIAL IDENTITIES AND EQUATIONS

KEY IDEAS

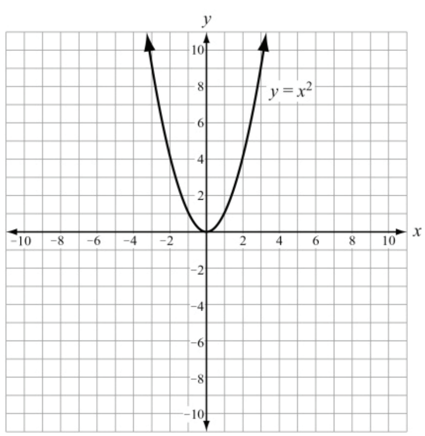
1. ***Quadratic equations*** are equations in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$. Solutions to quadratic equations are also called the ***roots*** of the equation. Real number solutions occur at the x-intercepts of the graph of the equation.

2. There are several methods to finding the solution(s) of a quadratic equation, including graphing, factoring, completing the square, and using the quadratic formula. Using the ***quadratic formula*** will produce real and complex solutions. The quadratic formula is $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

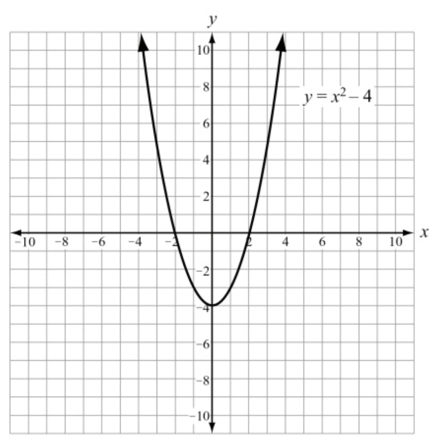
3. ***Complex solutions*** are in the form of $a + bi$, where a and b are real numbers.

Important Tip

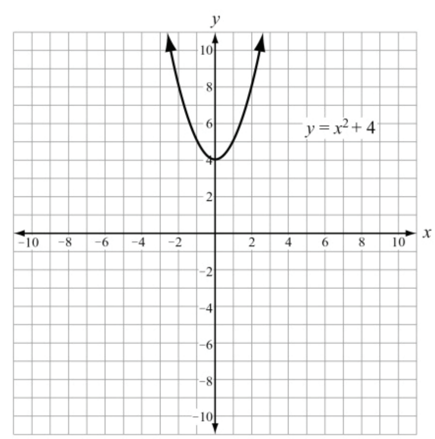
Complex solutions cannot be identified on the coordinate plane, because the graph will not have any x-intercepts. If an equation has complex solutions, they must be found algebraically. These graphs show quadratic functions with 1 real solution, 2 real solutions, and 2 complex solutions.



1 real solution



2 real solutions



2 complex solutions

REVIEW EXAMPLES

1) Solve $9x^2 + 16 = 0$ for x.

Solution:

$9x^2 + 16 = 0$	Original equation.
$9x^2 = -16$	Subtract 16 from both sides.
$x^2 = -\frac{16}{9}$	Divide both sides by 9.
$x = \pm\sqrt{-\frac{16}{9}}$	Take the square root of both sides.
$x = \pm\frac{4}{3}i$	Evaluate.

2) The function $h(t) = 4t^2 - 12t + 25$ represents the height, in inches, of a swing after t seconds, for $0 \le t \le 3$.

a. Solve the function when $h(t) = 0$.

b. Will the swing touch the ground? Explain how you know.

Solution:

a. Substitute 4 for a, –12 for b, and 25 for c in the quadratic formula.

$$t = \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(25)}}{2(4)}$$

$$t = \frac{12 \pm \sqrt{-256}}{8}$$

$$t = \frac{12 \pm 16i}{8}$$

$$t = \frac{3 \pm 4i}{2}$$

$$t = \frac{3}{2} \pm 2i$$

b. No; the quadratic equation has 2 imaginary roots, so there are no real number solutions. This means that the swing will not touch the ground. Furthermore, there are no real values of t that make $h(t)$ negative or 0.

EOCT Practice Items

1) What are the solutions to the equation $12x^2 = -300$?

A. $x = \pm 5$

B. $x = \pm 5i$

C. $x = 5 \pm i$

D. $x = -5 \pm i$

[Key: B]

2) What are the solutions to the equation $2x^2 + 3x + 9 = 0$?

A. $x = \frac{3}{4} \pm \frac{21}{4}i$

B. $x = -\frac{3}{4} \pm \frac{21}{4}i$

C. $x = \frac{3}{4} \pm \frac{3i\sqrt{7}}{4}$

D. $x = -\frac{3}{4} \pm \frac{3i\sqrt{7}}{4}$

[Key: D]

INTERPRET THE STRUCTURE OF EXPRESSIONS

KEY IDEAS

1. An ***algebraic expression*** contains variables, numbers, and operation symbols.

2. A ***term*** in an algebraic expression can be a constant, a variable, or a constant multiplied by a variable or variables. Every term is separated by a plus sign or minus sign.

   Example:

   The terms in the expression $5x^2 - 3x + 8$ are $5x^2$, $-3x$, and 8.

3. A ***coefficient*** is the constant number that is multiplied by a variable in a term.

   Example:

   The coefficient in the term $7x^2$ is 7.

4. The ***degree*** of an expression in one variable is the greatest exponent in the expression.

   Example:

   The degree of the expression $n^3 - 4n^2 + 7$ is 3.

5. A ***common factor*** is a variable or number that terms can by divided by without a remainder.

   Example:

   The common factors of $30x^2$ and $6x$ are 1, 2, 3, 6, and x.

6. A ***common factor of an expression*** is a number or term that the entire expression can be divided by without a remainder.

   Example:

   The common factor for the expression $3x^3 + 6x^2 - 15x$ is $3x$
   (because $3x^3 + 6x^2 - 15x = 3x(x^2 + 2x - 5)$)

7. If parts of an expression are independent of each other, the expression can be interpreted in different ways.

Example:

In the expression $\frac{1}{2}h(b_1 + b_2)$, the factors h and $(b_1 + b_2)$ are independent of each other. It can be interpreted as the product of h and a term that does not depend on h.

8. The structure of some expressions can be used to help rewrite them. For example, some fourth-degree expressions are in quadratic form.

Example:

$x^4 + 5x^2 + 4 = (x^2 + 4)(x^2 + 1)$

Example:

$$\begin{aligned} x^4 - y^4 &= \left(x^2\right)^2 - \left(y^2\right)^2 \\ &= \left(x^2 - y^2\right)\left(x^2 + y^2\right) \\ &= (x + y)(x - y)\left(x^2 + y^2\right) \end{aligned}$$

REVIEW EXAMPLES

1) Consider the expression $3n^2 + n + 2$.

   a. What is the coefficient of n?
   b. What terms are being added in the expression?

Solution:

   a. 1
   b. $3n^2$, n, and 2

2) Factor the expression $16a^4 - 81$.

Solution:

The expression $16a^4 - 81$ is quadratic in form, because it is the difference of two squares $\left(16a^4 = \left(4a^2\right)^2 \text{ and } 81 = 9^2\right)$ and both terms of the binomial are perfect squares. The difference of squares can be factored as:

$x^2 - y^2 = (x + y)(x - y)$.

$16a^4 - 81$	Original expression.
$(4a^2 + 9)(4a^2 - 9)$	Factor the binomial (difference of two squares).
$(4a^2 + 9)(2a + 3)(2a - 3)$	Factor $4a^2 - 9$ (difference of two squares).

3) Factor the expression $12x^4 + 14x^2 - 6$.

Solution:

$12x^4 + 14x^2 - 6$	Original expression.
$2(6x^4 + 7x^2 - 3)$	Factor the trinomial (common factor).
$2(3x^2 - 1)(2x^2 + 3)$	Factor.

EOCT Practice Items

1) In which expression is the coefficient of the *n* term –1?

A. $3n^2 + 4n - 1$

B. $-n^2 + 5n + 4$

C. $-2n^2 - n + 5$

D. $4n^2 + n - 5$

[Key: C]

2) Which expression is equivalent to $121x^4 - 64y^6$?

A. $(11x^2 - 16y^2)(11x^2 + 16y^2)$

B. $(11x^2 - 16y^3)(11x^2 - 16y^3)$

C. $(11x^2 + 8y^2)(11x^2 + 8y^2)$

D. $(11x^2 + 8y^3)(11x^2 - 8y^3)$

[Key: D]

3) The expression s^2 is used to calculate the area of a square, where *s* is the side length of the square. What does the expression $(8x)^2$ represent?

A. the area of a square with a side length of 8

B. the area of a square with a side length of 16

C. the area of a square with a side length of $4x$

D. the area of a square with a side length of $8x$

[Key: D]

WRITE EXPRESSIONS IN EQUIVALENT FORMS TO SOLVE PROBLEMS

KEY IDEAS

1. The ***zeros*** of a function are the values of the variable that make the function equal to zero. When the function is written in factored form, the zero product property can be used to find the zeros of the function. The zero product property states that if the product of two factors is zero, then one or both of the factors must be zero. So, the zeros of the function are the values that make either factor equal to zero.

   Example:

   $x^2 - 7x + 12 = 0$ Original equation.
   $(x-3)(x-4) = 0$ Factor.

   Set each factor equal to zero and solve.

$x - 3 = 0$	$x - 4 = 0$
$x = 3$	$x = 4$

   The zeros of the function $y = x^2 - 7x + 12$ are $x = 3$ and $x = 4$.

2. To ***complete the square*** of a quadratic function means to write a function as the square of a sum. The standard form for a quadratic expression is $ax^2 + bx + c$, where $a \neq 0$. When $a = 1$, completing the square of the function $x^2 + bx = d$ gives $\left(x + \frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$.

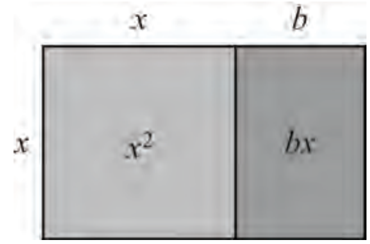
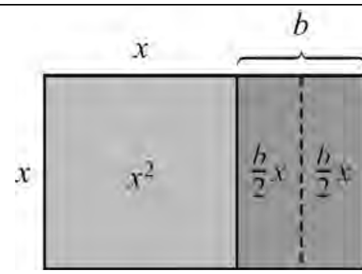
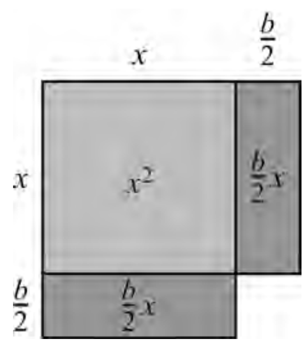
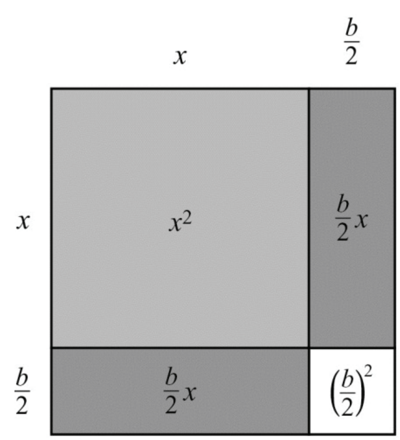
   To complete the square when the value $a \neq 1$, factor the value of a from the expression.

   Example:

   To complete the square, take half of the coefficient of the x-term, square it, and add it to both sides of the equation.

$x^2 + bx = d$	Original expression.
$x^2 + bx + \left(\frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$	The coefficient of x is b. Half of b is $\frac{b}{2}$. Add the square of $\frac{b}{2}$ to both sides of the equation.
$\left(x + \frac{b}{2}\right)^2 = d + \left(\frac{b}{2}\right)^2$	The expression on the left side of the equation is a perfect square trinomial. Factor to write it as a binomial squared.

This figure shows how a model can represent completing the square of the expression $x^2 + bx$, where b is positive.

This model represents the expression $x^2 + bx$. To complete the square, create a model that is a square.	
Split the rectangle for bx into two rectangles that represent $\frac{b}{2}x$.	
Rearrange the two rectangles that represent $\frac{b}{2}x$.	
The missing piece of the square measures $\frac{b}{2}$ by $\frac{b}{2}$. Add and subtract $\left(\frac{b}{2}\right)^2$ to complete the model of the square. The large square has a side length of $x+\frac{b}{2}$, so this model represents $\left(x+\frac{b}{2}\right)^2$. $x^2+bx=\left(x+\frac{b}{2}\right)^2-\left(\frac{b}{2}\right)^2$.	

Important Tip

When you complete the square, make sure you are only changing the form of the expression, and not changing the value.

- When completing the square in an expression, add **and subtract** half of the coefficient of the *x* term squared.
- When completing the square in an equation, add half of the coefficient of the *x* term squared **to both sides of the equation.**

Examples:

Complete the square –

$$x^2 + 3x + 7$$

$$\left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) + 7 - \left(\frac{3}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 + \frac{19}{4}$$

Complete the square –

$$x^2 + 3x + 7 = 0$$

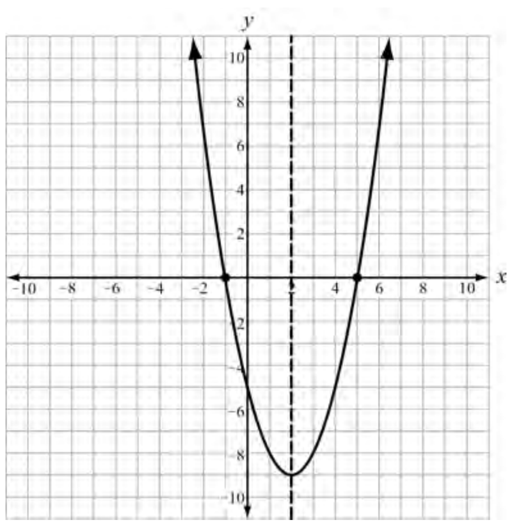
$$x^2 + 3x + \left(\frac{3}{2}\right)^2 = -7 + \left(\frac{3}{2}\right)^2$$

$$\left(x + \frac{3}{2}\right)^2 = -\frac{19}{4}$$

3. Every quadratic function has a ***minimum*** or a ***maximum***. This minimum or maximum is located at the ***vertex*** (h, k). The vertex (h, k) also identifies the ***axis of symmetry*** and the minimum or maximum value of the function. The axis of symmetry is $x = h$.

Example:

The quadratic equation $f(x) = x^2 - 4x - 5$ is shown in this graph. The minimum of the function occurs at the vertex (2, –9). The zeros of the function are (–1, 0) and (5, 0). The axis of symmetry is $x = 2$.



4. The ***vertex form*** of a quadratic function is $f(x) = a(x-h)^2 + k$, where (h, k) is the vertex. One way to convert an equation from standard form to vertex form is to complete the square.

5. The vertex of a quadratic function can also be found by using the ***standard form*** and determining the value $\frac{-b}{2a}$. The vertex is $\left(\frac{-b}{2a},\ f\left(\frac{-b}{2a}\right)\right)$.

REVIEW EXAMPLES

1) Write $f(x) = 2x^2 + 6x + 1$ in vertex form.

Solution:

The function is in standard form, where $a = 2$, $b = 6$, and $c = 1$.

$2x^2+6x+1$	Original expression.
$2(x^2+3x)+1$	Factor out 2 from the quadratic and linear terms.
$2\left(x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2\right)+1$	Add and subtract the square of half of the coefficient of the linear term.
$2\left(x^2+3x+\left(\frac{3}{2}\right)^2\right)-\frac{9}{2}+1$	Remove the subtracted term from the parentheses. Remember to multiply by *a*.
$2\left(x^2+3x+\left(\frac{3}{2}\right)^2\right)-\frac{7}{2}$	Combine the constant terms.
$2\left(x+\frac{3}{2}\right)^2-\frac{7}{2}$	Write the perfect square trinomial as a binomial squared.

The vertex of the function is $\left(-\frac{3}{2},-\frac{7}{2}\right)$.

The vertex of a quadratic function can also be found by writing the polynomial in standard form and determining the value of $\frac{-b}{2a}$. The vertex is $\left(\frac{-b}{2a},\ f\left(\frac{-b}{2a}\right)\right)$.

For $f(x) = 2x^2 + 6x + 1$, $a = 2$, $b = 6$, and $c = 1$.

$$\frac{-b}{2a}=\frac{-6}{2(2)}=\frac{-6}{4}=\frac{-3}{2}$$

$$f\left(-\frac{3}{2}\right) = 2\left(-\frac{3}{2}\right)^2 + 6\left(-\frac{3}{2}\right) + 1$$
$$= 2\left(\frac{9}{4}\right) - \frac{18}{2} + 1$$
$$= \frac{9}{2} - 9 + 1$$
$$= -\frac{7}{2}$$

The vertex of the function is $\left(-\frac{3}{2}, -\frac{7}{2}\right)$.

2) The function $h(t) = -t^2 + 8t + 2$ represents the height, in feet, of a stream of water being squirted out of a fountain after t seconds. What is the maximum height of the water?

Solution:

The function is in standard form, where $a = -1$, $b = 8$, and $c = 2$.

The x-coordinate of the vertex is $\frac{-b}{2a} = \frac{-8}{2(-1)} = 4$.

The y-coordinate of the vertex is $h(4) = -(4)^2 + 8(4) + 2 = 18$

The vertex of the function is (4, 18). So, the maximum height of the water is 18 feet.

3) What are the zeros of the function represented by the quadratic expression $x^2 + 6x - 27$?

Solution:

Factor the expression: $x^2 + 6x - 27 = (x + 9)(x - 3)$.

Set each factor equal to 0 and solve for x.

$$x + 9 = 0 \qquad\qquad x - 3 = 0$$
$$x = -9 \qquad\qquad x = 3$$

The zeros are $x = -9$ and $x = 3$.

EOCT Practice Items

1) What are the zeros of the function represented by the quadratic expression $2x^2 + x - 3$?

A. $x = -\frac{3}{2}$ and $x = 1$

B. $x = -\frac{2}{3}$ and $x = 1$

C. $x = -1$ and $x = \frac{2}{3}$

D. $x = -1$ and $x = -\frac{3}{2}$

[Key: A]

2) What is the vertex of the graph of $f(x) = x^2 + 10x - 9$?

A. (5, 66)

B. (5, –9)

C. (–5, –9)

D. (–5, –34)

[Key: D]

3) Which of the following is the result of completing the square for the expression $x^2 + 8x - 30$?

A. $(x + 4)^2 - 30$

B. $(x + 4)^2 - 46$

C. $(x + 8)^2 - 30$

D. $(x + 8)^2 - 94$

[Key: B]

4) The expression $-x^2 + 70x - 600$ represents a company's profit for selling x items. For which number(s) of items sold is the company's profit equal to $0?

A. 0 items

B. 35 items

C. 10 items and 60 items

D. 20 items and 30 items

[Key: C]

CREATE EQUATIONS THAT DESCRIBE NUMBERS OR RELATIONSHIPS

KEY IDEAS

1. Quadratic equations and inequalities can be written to model real-world situations. A quadratic equation can have 0, 1, or 2 real solutions. A quadratic inequality has a set of solutions.

   Here are some examples of real-world situations that can be modeled by quadratic functions:

   - Finding the area of a shape: Given that the length of a rectangle is 5 units more than the width, the area of the rectangle in square units can be represented by $A = x(x + 5)$.
   - Finding the product of consecutive integers: Given a number, n, the next consecutive number is $n + 1$ and the next consecutive even (or odd) number is $n + 2$. The product, P, of two consecutive numbers is $P = n(n + 1)$.
   - Finding the height of a projectile that is thrown, shot, or dropped: When heights are given in metric units, the equation used is $h(t) = -4.9t^2 + v_o t + h_o,$ where v_o is the initial velocity and h_o is the initial height, in meters. The coefficient –4.9 represents half the force of gravity. When heights are given in customary units, the equation used is $h(t) = -16t^2 + v_o t + h_o,$ where v_o is the initial velocity and h_o is the initial height, in feet. The coefficient –16 represents half the force of gravity. For example, the height, in feet, of a ball thrown with an initial velocity of 60 feet per second and an initial height of 4 feet can be represented by $h(t) = -16t^2 + 60t + 4$, where t is seconds.

   In each example, a quadratic inequality can be formed by using inequality symbols in place of the equal sign. For example, the product of two consecutive numbers that is less than 30 can be represented by the quadratic inequality $n(n + 1) < 30$.

2. You can use the properties of equality to solve for a variable in an equation. Use inverse operations on both sides of the equation until you have isolated the variable.

   Example:

   Solve $S = 2\pi r^2 + 2\pi rh$ for h.

Solution:

First, subtract $2\pi r^2$ from both sides. Then divide both sides by $2\pi r$.

$$S = 2\pi r^2 + 2\pi rh$$

$$S - 2\pi r^2 = 2\pi rh$$

$$\frac{S - 2\pi r^2}{2\pi r} = h$$

$$\frac{S}{2\pi r} - r = h$$

3. To graph a quadratic equation, find the vertex of the graph and the zeros of the equation. The axis of symmetry goes through the vertex, and divides the graph into two sides that are mirror images of each other. To draw the graph, you can plot points on one side of the parabola and use symmetry to find the corresponding points on the other side of the parabola.

4. The axis of symmetry is the midpoint for each corresponding pair of *x*-coordinates with the same *y*-value. If(x_1, y) is a point on the graph of a parabola and $x = h$ is the axis of symmetry, then (x_2, y) is also a point on the graph, and x_2 can be found using this equation: $\frac{x_1 + x_2}{2} = h$.

REVIEW EXAMPLES

1) The product of two consecutive positive integers is 132.

   a. Write an equation to model the situation.
   b. What are the two consecutive integers?

Solution:

a. Let n represent the lesser of the two integers. Then $n + 1$ represents the greater of the two integers. So, the equation is $n(n + 1) = 132$.

b. Solve the equation for n.

$n(n + 1) = 132$	Original equation.
$n^2 + n = 132$	Distributive Property.
$n^2 + n - 132 = 0$	Subtract 132 from both sides.
$(n + 12)(n - 11) = 0$	Factor.

Set each factor equal to 0 and solve for n.

$n + 12 = 0$	$n - 11 = 0$
$n = -12$	$n = 11$

Because the two consecutive integers are both positive, $n = -12$ cannot be the solution. So, $n = 11$ is the solution, which means that the two consecutive integers are 11 and 12.

2) The formula for the volume of a cylinder is $V = \pi r^2 h$.

   a. Solve the formula for r.
   b. If the volume of a cylinder is 200π cubic inches and the height of the cylinder is 8 inches, what is the radius of the cylinder?

Solution:

a. Solve the formula for r.

$V = \pi r^2 h$	Original formula.
$\dfrac{V}{\pi h} = r^2$	Divide both sides by πh.
$\pm\sqrt{\dfrac{V}{\pi h}} = r$	Take the square root of both sides.
$\sqrt{\dfrac{V}{\pi h}} = r$	Choose the positive value because the radius cannot be negative.

b. Substitute 200π for *V* and 8 for *h*, and evaluate.

$$r = \sqrt{\frac{V}{\pi h}} = \sqrt{\frac{200\pi}{\pi(8)}} = \sqrt{\frac{200}{8}} = \sqrt{25} = 5$$

The radius of the cylinder is 5 inches.

3) Graph the function represented by the equation $y = 3x^2 - 6x - 9$.

Solution:

Find the zeros of the equation.

$0 = 3x^2 - 6x - 9$	Set the equation equal to 0.
$0 = 3(x^2 - 2x - 3)$	Factor out 3.
$0 = 3(x - 3)(x + 1)$	Factor.
$0 = (x - 3)(x + 1)$	Divide both sides by 3.

Set each factor equal to 0 and solve for *x*.

$x - 3 = 0$	$x + 1 = 0$
$x = 3$	$x = -1$

The zeros are at $x = -1$ and $x = 3$.

Find the vertex of the graph.

$$\frac{-b}{2a} = \frac{-(-6)}{2(3)} = \frac{6}{6} = 1$$

Substitute 1 for *x* in the original equation to find the *y*-value of the vertex:

$$\begin{aligned} 3(1)^2 - 6(1) - 9 &= 3 - 6 - 9 \\ &= -12 \end{aligned}$$

Graph the two *x*-intercepts (3, 0), (–1, 0), and the vertex (1, –12).

Find the *y*-intercept by substituting 0 for *x*.

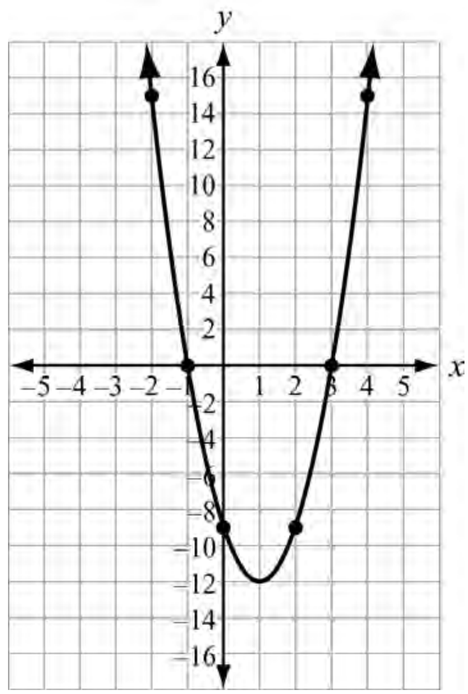
$y = 3x^2 - 6x - 9$
$y = 3(0)^2 - 6(0) - 9$
$y = -9$

Find $f(-2)$ by substituting –2 for *x*.

$y = 3x^2 - 6x - 9$
$y = 3(-2)^2 - 6(-2) - 9$
$y = 3(4) + 12 - 9$
$y = 15$

Graph the points (0, –9) and (–2, 15). Then use the concept of symmetry to draw the rest of the function. The axis of symmetry is $x = 1$.

- (0, –9) is a point on the graph; $\frac{0+x}{2}=1$; $x=2$; so (2, –9) is also a point on the graph.
- (–2, 15) is a point on the graph; $\frac{-2+x}{2}=1$; $x=4$; so (4, 15) is also a point on the graph.



EOCT Practice Items

1) **A garden measuring 8 feet by 12 feet will have a walkway around it. The walkway has a uniform width, and the area covered by the garden and the walkway is 192 square feet. What is the width of the walkway?**

   A. 2 feet

   B. 3.5 feet

   C. 4 feet

   D. 6 feet

[Key: A]

2) **The formula for the surface area of a cone is $SA = \pi r^2 + \pi rs$. Which equation shows the formula in terms of *s*?**

   A. $s = \frac{SA}{\pi r} - \pi r^2$

   B. $s = \frac{SA}{\pi r} + \pi r^2$

   C. $s = \frac{SA - \pi r^2}{\pi r}$

   D. $s = \frac{SA + \pi r^2}{\pi r}$

[Key: C]

SOLVE EQUATIONS AND INEQUALITIES IN ONE VARIABLE

KEY IDEAS

1. When quadratic equations do not have a linear term, you can solve the equation by taking the ***square root*** of each side of the equation. This method provides rational and irrational values for x, as well as complex solutions. Remember, every square root has a positive value and a negative value. Earlier in the guide, we eliminated the negative answers when they represented length or distance.

Examples:

$$\begin{aligned} 3x^2 - 147 &= 0 \\ 3x^2 &= 147 \\ x^2 &= 49 \\ x &= \pm 7 \end{aligned}$$

$$\begin{aligned} 3x^2 + 147 &= 0 \\ 3x^2 &= -147 \\ x^2 &= -49 \\ x &= \pm 7i \end{aligned}$$

Check your answers:

$$\begin{aligned} 3(7)^2 - 147 &= 3(49) - 147 \\ &= 147 - 147 \\ &= 0 \end{aligned}$$

$$\begin{aligned} 3(7i)^2 + 147 &= 3(-49) + 147 \\ &= -147 + 147 \\ &= 0 \end{aligned}$$

$$\begin{aligned} 3(-7)^2 - 147 &= 3(49) - 147 \\ &= 147 - 147 \\ &= 0 \end{aligned}$$

$$\begin{aligned} 3(-7i)^2 + 147 &= 3(-49) + 147 \\ &= -147 + 147 \\ &= 0 \end{aligned}$$

2. You can ***factor*** some quadratic equations to find the solutions. To do this, rewrite the equation in standard form set equal to zero ($ax^2 + bx + c = 0$). Factor the expression, set each factor to 0 (by the Zero Product Property), and then solve for x in each resulting equation. This will provide two rational values for x.

Example:

$$\begin{aligned} x^2 - x &= 12 \\ x^2 - x - 12 &= 0 \\ (x-4)(x+3) &= 0 \end{aligned}$$

Set each factor equal to 0 and solve.

$$x - 4 = 0 \qquad x + 3 = 0$$
$$x = 4 \qquad x = -3$$

Check your answers:

$$4^2 - 4 = 16 - 4 \qquad (-3)^2 - (-3) = 9 + 3$$
$$= 12 \qquad = 12$$

3. You can complete the square to solve a quadratic equation. First, write the expression that represents the function in standard form, $ax^2 + bx + c = 0$. Subtract the constant from both sides of the equation: $ax^2 + bx = -c$. Divide both sides of the equation by a: $x^2 + \frac{b}{a}x = \frac{-c}{a}$. Add the square of half the coefficient of the x-term to both sides: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$. Write the perfect square trinomial as a binomial squared: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$. Take the square root of both sides of the equation and solve for x.

4. All quadratic equations can be solved using the quadratic formula. The ***quadratic formula*** is $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, where $ax^2 + bx + c = 0$. The quadratic formula will yield both real and complex solutions of the equation.

Important Tip

While there may be several methods that can be used to solve a quadratic equation, some methods may be easier than others for certain equations.

REVIEW EXAMPLES

1) The standard form of a quadratic equation is $ax^2 + bx + c = 0$.

   a. After subtracting c from both sides of the equation, what would you add to both sides of the equation to complete the square?

   b. Solve for x. What formula did you derive?

Solution:

a.

$ax^2 + bx + c = 0$	Original equation.
$ax^2 + bx = -c$	Subtract the constant c from both sides of the equation.
$x^2 + \frac{b}{a}x = \frac{-c}{a}$	Divide both sides of the equation by a.
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$	Add the square of half of the coefficient of the x term.

To complete the square, you would add $\left(\frac{b}{2a}\right)^2$ to both sides of the equation.

b. Solve for x.

$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-c}{a} + \left(\frac{b}{2a}\right)^2$	
$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{-4ac}{4a^2} + \frac{b^2}{4a^2}$	Rewrite the right side of the equation using the common denominator of $4a^2$.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Factor the left side of the equation and express the right side as a single fraction.
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Take the square root of both sides of the equation.
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Subtract $\frac{b}{2a}$ from both sides of the equation.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Combine like terms.

This is the quadratic formula.

2) Solve the equation $x^2 - 10x + 25 = 0$ by factoring.

Solution:

Factor the equation $(x - 5)(x - 5)$.
Both factors are the same, so solve the equation:
$x - 5 = 0$
$x = 5$

3) Solve the equation $x^2 - 100 = 0$ by using square roots.

Solution:

Solve the equation using square roots.

$x^2 = 100$	Add 100 to both sides of the equation.
$x = \pm\sqrt{100}$	Take the square root of both sides of the equation.
$x = \pm 10$	Evaluate.

EOCT Practice Items

1) What are the solutions to the equation $2x^2 - 2x - 12 = 0$?

A. $x = -4, x = 3$

B. $x = -3, x = 4$

C. $x = -2, x = 3$

D. $x = -6, x = 2$

[Key: C]

2) What are the solutions to the equation $4x^2 + 8x + 20 = 0$?

A. $x = 1 \pm 2i$

B. $x = -1 \pm 2i$

C. $x = 1 \pm i$

D. $x = -1 \pm i$

[Key: B]

3) What are the solutions to the equation $6x^2 - x - 40 = 0$?

A. $x = -\frac{8}{3},\ x = -\frac{5}{2}$

B. $x = -\frac{8}{3},\ x = \frac{5}{2}$

C. $x = \frac{5}{2},\ x = \frac{8}{3}$

D. $x = -\frac{5}{2},\ x = \frac{8}{3}$

[Key: D]

4) What are the solutions to the equation $x^2 - 5x = 14$?

A. $x = -7$, $x = -2$

B. $x = -14$, $x = -1$

C. $x = -2$, $x = 7$

D. $x = -1$, $x = 14$

[Key: C]

5) An object is thrown in the air with an initial velocity of 5 m/s from a height of 9 m. The equation $h(t) = -4.9t^2 + 5t + 9$ models the height of the object in meters after t seconds.

How many seconds does it take for the object to hit the ground?

A. 0.94 seconds

B. 1.77 seconds

C. 1.96 seconds

D. 9.0 seconds

[Key: C]

SOLVE SYSTEMS OF EQUATIONS

KEY IDEAS

1. A ***system of equations*** is a collection of equations that have the same variables. A system of equations can be solved either algebraically or graphically.

2. To algebraically solve a system of equations involving a linear equation and a quadratic equation, first solve the linear equation for a variable. Then, substitute into the quadratic equation. Once you have found the solution for one variable, substitute the value into the other equation and solve for the second variable.

Example:

$$\begin{cases} y = x^2 + 2x - 9 \\ x - y = 3 \end{cases}$$

First, solve the second equation for y.

$x - y = 3$ Original equation.
$x - 3 = y$

Because both equations are solved for the same variable, substitute $x - 3$ for y in the quadratic equation and solve for x.

$$x^2 + 2x - 9 = x - 3$$
$$x^2 + x - 6 = 0$$
$$(x+3)(x-2) = 0$$
$$x + 3 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -3 \quad \text{or} \quad x = 2$$

Substitute the x-values into one of the equations to solve for the corresponding y-values.

$$x - y = 3 \qquad\qquad x - y = 3$$
$$-3 - y = 3 \qquad\qquad 2 - y = 3$$
$$y = -6 \qquad\qquad y = -1$$

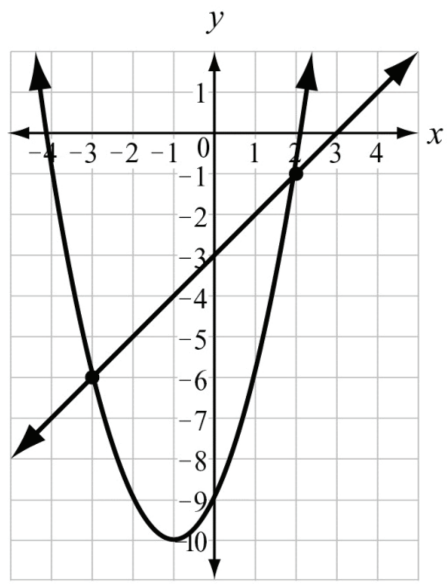
The solutions are $(-3, -6)$ and $(2, -1)$.

3. To graphically solve a system of equations involving a linear equation and a quadratic equation, graph both equations on the same coordinate plane. The point (or points) of intersection are the solutions.

Example:

For the system of equations given, graph the equations on a coordinate plane.

$$y = x^2 + 2x - 9$$
$$x - y = 3$$



The solutions appear to be (–3, –6) and (2, –1).

Important Tip

Solving a system of equations graphically will identify the approximate solutions. Solving algebraically will produce the exact solutions of the system. If you solve a system graphically, it is necessary to check your solutions algebraically by substituting them into both original equations.

REVIEW EXAMPLES

1) What are the solutions of this system of equations?

$$y = -3x^2 + 4x + 20$$
$$y = -2x - 4$$

Solution:

In this example, both the linear and quadratic equations are already solved for the same variable, y. So, set the equations equal to each other and move all the terms to one side of the equation.

$$-3x^2 + 4x + 20 = -2x - 4$$
$$-3x^2 + 6x + 24 = 0$$

Factor the resulting equation and solve for x.

$$-3x^2 + 6x + 24 = 0$$
$$-3(x^2 - 2x - 8) = 0$$
$$x^2 - 2x - 8 = 0$$
$$(x - 4)(x + 2) = 0$$

$x - 4 = 0$ or $x + 2 = 0$
$x = 4$ or $x = -2$

Substitute –2 and 4 for one of the equations to find the y-values.

$y = -2(-2) - 4 = 4 - 4 = 0$
$y = -2(4) - 4 = -8 - 4 = -12$

The solutions to the system are (–2, 0) and (4, –12).

2) What are the solutions of this system of equations?

$$y = -2x^2 + 32$$
$$y = 4x + 2$$

Solution:

Graph both equations and identify the points of intersection.

First, graph $y = -2x^2 + 32$.

Find the vertex using $\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$, with $a = 2$ and $b = 0$.

$$\frac{-b}{2a} = \frac{0}{2(2)} = 0$$

$f(0) = -2(0)^2 + 32 = 32$

The vertex is (0, 32).

Find the zeros by setting $y = 0$.

$0 = -2x^2 + 32$	Set $y = 0$.
$0 = -2(x - 4)(x + 4)$	Factor.
$0 = (x - 4)(x + 4)$	Divide both sides by –2.

Set each factor equal to zero and solve for x.

$x - 4 = 0$ $\quad$ $x + 4 = 0$

$x = 4$ $\quad$ $x = -4$

The zeros are 4 and –4.

Graph the points (0, 32), (–4, 0), and (4, 0).

$f(-5) = -2(-5)^2 + 32 = -18$

Graph (–5, –18). Using symmetry, (5, –18) is also a point on the graph.

Draw a smooth curve to sketch the graph of the parabola.

Now graph $y = 4x + 2$.

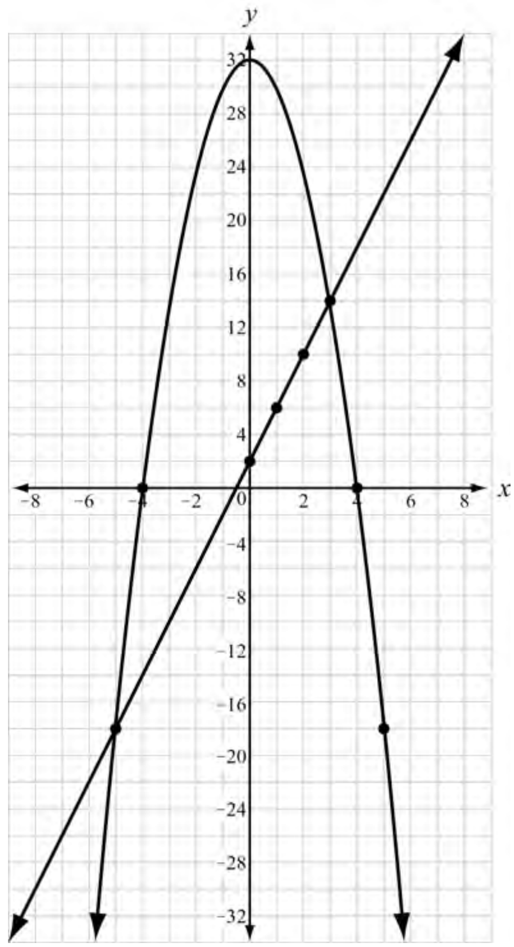
Find the y-intercept by setting $x = 0$.

$y = 4(0) + 2$

$y = 2$

The y-intercept is 2. Graph the point (0, 2). Use the slope of 4 to graph the points (1, 6) and (2, 10).

The graphs of the equations appear to intersect at (–5, –18) and (3, 14). You can verify algebraically that these points are the solutions of the system of equations.



Check your answers:

$-18 = -2(-5)^2 + 32$	$14 = -2(3)^2 + 32$
$-18 = -2(25) + 32$	$14 = -2(9) + 32$
$-18 = -50 + 32$	$14 = 14$
$-18 = -18$	

$-18 = 4(-5) + 2$	$14 = 4(3) + 2$
$-18 = -20 + 2$	$14 = 12 + 2$
$-18 = -18$	$14 = 14$

EOCT Practice Items

1) What are the solutions of this system of equations?

$$y = 5x^2 + 7x - 6$$
$$y = 12x - 6$$

A. (0, 6) and (1, –6)

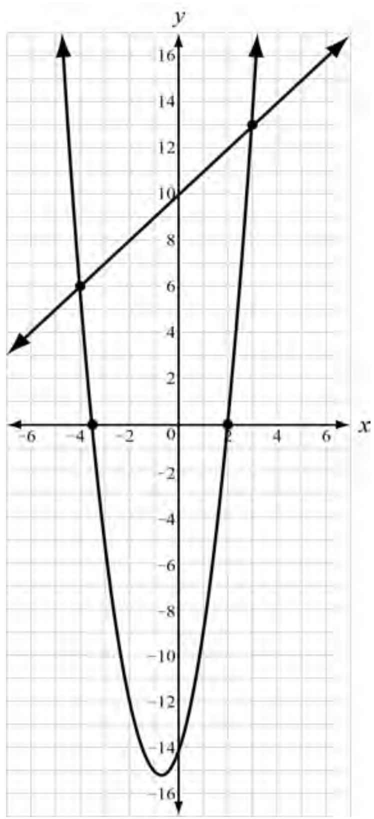
B. (0, –6) and (1, –6)

C. (0, –6) and (1, 6)

D. (–6, 0) and (6, 1)

[Key: C]

2) What appear to be the solutions of the system of equations shown in the graph?



A. (4, 6) and (3, 13)

B. (–4, 6) and (3, 13)

C. (–4, 13) and (3, 6)

D. (–3, 13) and (4, 6)

[Key: B]

INTERPRET FUNCTIONS THAT ARISE IN APPLICATIONS IN TERMS OF THE CONTEXT

KEY IDEAS

1. An ***x*-intercept** of a function is the *x*-coordinate of a point where the function crosses the *x*-axis. A function may have multiple *x*-intercepts. To find the *x*-intercepts of a quadratic function, set the function equal to 0 and solve for *x*. This can be done by factoring, completing the square, or using the quadratic formula.

2. The ***y*-intercept** of a function is the *y*-coordinate of the point where the function crosses the *y*-axis. A function may have zero or one *y*-intercepts. To find the *y*-intercept of a quadratic function, find the value of the function when *x* equals 0.

3. A function is ***increasing*** over an interval when the values of *y* increase as the values of *x* increase over that interval.

4. A function is ***decreasing*** over an interval when the values of *y* decrease as the values of *x* increase over that interval.

5. Every quadratic function has a ***minimum*** or ***maximum***, which is located at the vertex. When the function is written in standard form, the *x*-coordinate of the vertex is $\frac{-b}{2a}$. To find the *y*-coordinate of the vertex, substitute the value of $\frac{-b}{2a}$ into the function and evaluate.

6. The ***end behavior*** of a function describes how the values of the function change as the *x*-values approach negative infinity and positive infinity.

7. The ***domain*** of a function is the set of values for which it is possible to evaluate the function. The domain of a quadratic function is typically all real numbers, although in real-world applications it may only make sense to look at the domain values on a particular interval. For example, time must be a nonnegative number.

8. The ***average rate of change*** of a function over a specified interval is the slope of the line that connects the endpoints of the function for that interval. To calculate the average rate of change of a function over the interval from *a* to *b*, evaluate the expression $\frac{f(b)-f(a)}{b-a}$.

REVIEW EXAMPLES

1) A ball is thrown into the air from a height of 4 feet at time $t = 0$. The function that models this situation is $h(t) = -16t^2 + 63t + 4$, where t is measured in seconds and h is the height in feet.

   a. What is the height of the ball after 2 seconds?
   b. When will the ball reach a height of 50 feet?
   c. What is the maximum height of the ball?
   d. When will the ball hit the ground?
   e. What domain makes sense for the function?

Solution:

a. To find the height of the ball after 2 seconds, substitute 2 for t in the function.
$h(2) = -16(2)^2 + 63(2) + 4 = -16(4) + 126 + 4 = -64 + 126 + 4 = 66$
The height of the ball after 2 seconds is 66 feet.

b. To find when the ball will reach a height of 50 feet, find the value of t that makes $h(t) = 50$.
$50 = -16t^2 + 63t + 4$
$0 = -16t^2 + 63t - 46$

Use the quadratic formula with $a = -16$, $b = 63$, and $c = -46$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-63 \pm \sqrt{(63)^2 - 4(-16)(-46)}}{2(-16)}$$

$$t = \frac{-63 \pm \sqrt{3969 - 2944}}{-32}$$

$$t = \frac{-63 \pm \sqrt{1025}}{-32}$$

$t \approx 0.97$ or $t \approx 2.97$. So, the ball is at a height of 50 feet after approximately 0.97 seconds and 2.97 seconds.

c. To find the maximum height, find the vertex of $h(t)$.

The x-coordinate of the vertex is equal to $\frac{-b}{2a}: \frac{-63}{2(-16)} \approx 1.97.$ To find the y-coordinate, find $h(1.97)$:

$$h(1.97) = -16(1.97)^2 + 63(1.97) + 4$$
$$\approx 66$$

The maximum height of the ball is about 66 feet.

d. To find when the ball will hit the ground, find the value of t that makes $h(t) = 0$ (because 0 represents 0 feet from the ground).

$0 = -16t^2 + 63t + 4$

Using the quadratic formula (or by factoring), $t = -0.0625$ or $t = 4$.

Time cannot be negative, so $t = -0.0625$ is not a solution. The ball will hit the ground after 4 seconds.

e. Time must always be nonnegative and can be expressed by any fraction or decimal. The ball is thrown at 0 seconds and reaches the ground after 4 seconds. So, the domain $0 \le t \le 4$ makes sense for function $h(t)$.

2) This table shows a company's profit, p, in thousands of dollars over time, t, in months.

Time, *t* (months)	**Profit, *p* (thousands of dollars)**
3	18
7	66
10	123
15	258
24	627

a. Describe the average rate of change in terms of the given context.
b. What is the average rate of change of the profit between 3 and 7 months?
c. What is the average rate of change of the profit between 3 and 24 months?

Solution:

a. The average rate of change represents the rate at which the company earns a profit.

b. Use the expression for average rate of change. Let $a = 3$, $b = 7$, $f(a) = 18$, and $f(b) = 66$.

$$\frac{f(b) - f(a)}{b - a} = \frac{66 - 18}{7 - 3} = \frac{48}{4} = 12$$

The average rate of change between 3 and 7 months is 12 thousand dollars ($12,000) per month.

c. Use the expression for average rate of change. Let $a = 3$, $b = 24$, $f(a) = 18$, and $f(b) = 627$.

$$\frac{f(b)-f(a)}{b-a}=\frac{627-18}{24-3}=\frac{609}{21}=29$$

The average rate of change between 3 and 24 months is 29 thousand dollars ($29,000) per month.

EOCT Practice Items

1) A flying disk is thrown into the air from a height of 25 feet at time $t = 0$. The function that models this situation is $h(t) = -16t^2 + 75t + 25$, where t is measured in seconds and h is the height in feet. What values of t best describe the times when the disk is flying in the air?

A. $0 < t < 5$

B. $0 < t < 25$

C. all real numbers

D. all positive integers

[Key: A]

2) Use this table to answer the question.

x	$f(x)$
–2	15
–1	9
0	5
1	3
2	3

What is the average rate of change of $f(x)$ over the interval $-2 \le f(x) \le 0$?

A. –10

B. –5

C. 5

D. 10

[Key: B]

3) What is the end behavior of the graph of $f(x) = -0.25x^2 - 2x + 1$?

A. As x increases, $f(x)$ increases.
As x decreases, $f(x)$ decreases.

B. As x increases, $f(x)$ decreases.
As x decreases, $f(x)$ decreases.

C. As x increases, $f(x)$ increases.
As x decreases, $f(x)$ increases.

D. As x increases, $f(x)$ decreases.
As x decreases, $f(x)$ increases. [Key: B]

ANALYZE FUNCTIONS USING DIFFERENT REPRESENTATIONS

KEY IDEAS

1. Functions can be represented algebraically, graphically, numerically (in tables), or verbally (by description).

Examples:

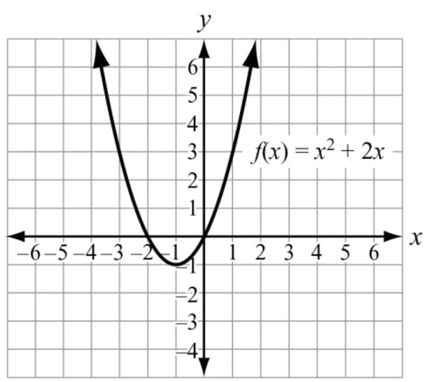
Algebraically: $f(x) = x^2 + 2x$

Verbally (by description): A function that represents the sum of the square of a number and twice the number.

Numerically (in a table):

x	***f*(*x*)**
–1	–1
0	0
1	3
2	8

Graphically:



2. You can compare key features of two functions represented in different ways. For example, if you are given an equation of a quadratic function, and a graph of another quadratic function, you can calculate the vertex of the first function and compare it to the vertex of the graphed function.

REVIEW EXAMPLES

1) Graph the function $f(x) = x^2 - 5x - 24$.

Solution:

Use the algebraic representation of the function to find the key features of the graph of the function.

Find the zeros of the function.

$0 = x^2 - 5x - 24$	Set the function equal to 0.
$0 = (x - 8)(x + 3)$	Factor.

Set each factor equal to 0 and solve for x.

$$x - 8 = 0 \qquad x + 3 = 0$$
$$x = 8 \qquad x = -3$$

The zeros are at $x = -3$ and $x = 8$.

Find the vertex of the function.

$$x = \frac{-b}{2a} = \frac{-(-5)}{2(1)} = \frac{5}{2} = 2.5$$

Substitute 2.5 for x in the original function to find $f(2.5)$:

$f(x) = x^2 - 5x - 24$
$f(2.5) = (2.5)^2 - 5(2.5) - 24 = 6.25 - 12.5 - 24 = -30.25$.

The vertex is (2.5, –30.25).

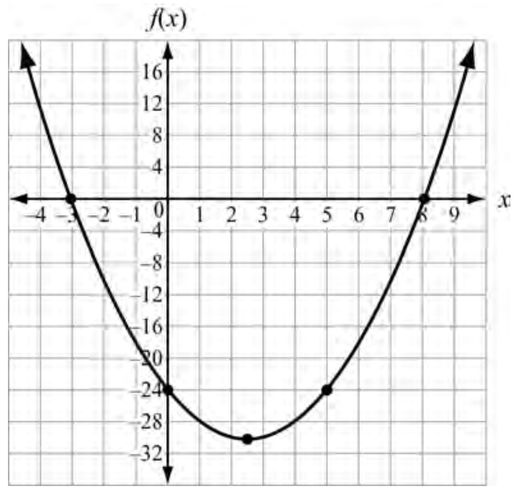
Find the y-intercept by finding $f(0)$.
$f(x) = x^2 - 5x - 24$
$f(0) = (0)^2 - 5(0) - 24 = -24$

The y-intercept is (0, –24). Use symmetry to find another point. The line of symmetry is $x = 2.5$.

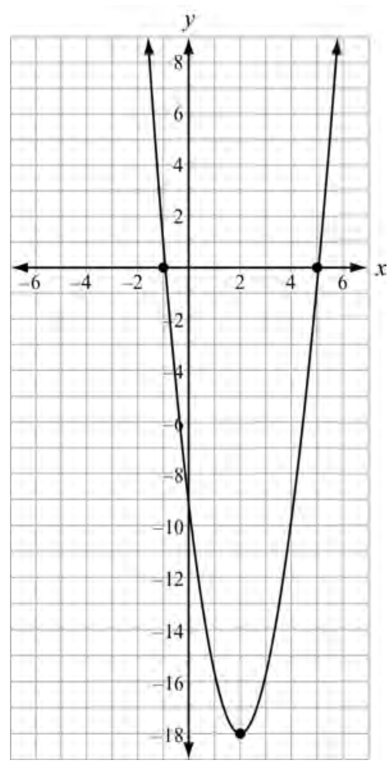
$$\frac{0 + x}{2} = 2.5$$
$$x = 5$$

So, point (5, –24) is also on the graph.

Plot the points (–3, 0), (8, 0), (2.5, –30.25), (0, –24) and (5, –24). Draw a smooth curve through the points.



2) This graph shows a function $f(x)$.



Compare the graph of $f(x)$ to the graph of the function given by the equation $g(x) = 4x^2 + 6x - 18$. Which function has the lesser minimum value? How do you know?

Solution:

The minimum value of a quadratic function is the *y*-value of the vertex.
The vertex of the graph of $f(x)$ appears to be (2, –18). So, the minimum value is –18.

Find the vertex of the function $g(x) = 4x^2 + 6x - 18$.

To find the vertex of $g(x)$, use $\left(\frac{-b}{2a}, g\left(\frac{-b}{2a}\right)\right)$ with $a = 4$ and $b = 6$.

$$x = \frac{-b}{2a} = \frac{-(6)}{2(4)} = \frac{-6}{8} = -0.75$$

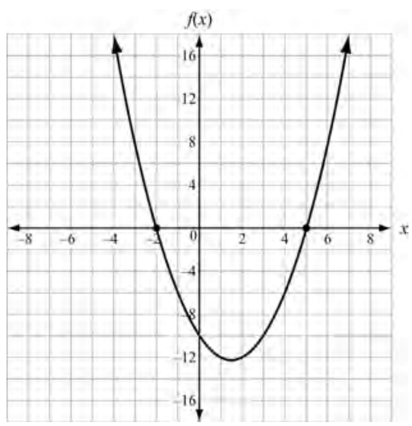
Substitute –0.75 for x in the original function $g(x)$ to find $g(-0.75)$:

$$\begin{aligned} g(x) &= 4x^2 + 6x - 18 \\ g(-0.75) &= 4(-0.75)^2 + 6(-0.75) - 18 \\ &= 2.25 - 4.5 - 18 \\ &= -20.25 \end{aligned}$$

The minimum value of $g(x)$ is –20.25.
–20.25 < –18, so the function $g(x)$ has the lesser minimum value.

EOCT Practice Items

1) Use this graph to answer the question.



Which function is shown in the graph?

A. $f(x) = x^2 - 3x - 10$

B. $f(x) = x^2 + 3x - 10$

C. $f(x) = x^2 + x - 12$

D. $f(x) = x^2 - 5x - 8$

[Key: A]

2) The function $f(t) = –16t^2 + 64t + 5$ models the height of a ball that was hit into the air, where t is measured in seconds and h is the height in feet.

This table represents the height, $g(t)$, of a second ball that was thrown into the air.

Time, t (in seconds)	**Height, $g(t)$ (in feet)**
0	4
1	36
2	36
3	4

Which statement BEST compares the length of time each ball is in the air?

A. The ball represented by $f(t)$ is in the air for about 5 seconds, and the ball represented by $g(t)$ is in the air for about 3 seconds.

B. The ball represented by $f(t)$ is in the air for about 3 seconds, and the ball represented by $g(t)$ is in the air for about 5 seconds.

C. The ball represented by $f(t)$ is in the air for about 3 seconds, and the ball represented by $g(t)$ is in the air for about 4 seconds.

D. The ball represented by $f(t)$ is in the air for about 4 seconds, and the ball represented by $g(t)$ is in the air for about 3 seconds.

[Key: D]

BUILD A FUNCTION THAT MODELS A RELATIONSHIP BETWEEN TWO QUANTITIES

KEY IDEAS

1. An ***explicit expression*** contains variables, numbers, and operation symbols, and does not use an equal sign to relate the expression to another quantity.

2. A ***recursive process*** can show that a quadratic function has second differences that are equal to one another.

   Example:

   Consider the function $f(x) = x^2 + 4x - 1$.

   This table of values shows five values of the function.

x	$f(x)$
–2	–5
–1	–4
0	–1
1	4
2	11

   The first and second differences are shown.

x	$f(x)$
–2	–5
–1	–4
0	–1
1	4
2	11

First differences

$-4 - (-5) = 1$
$-1 - (-4) = 3$
$4 - (-1) = 5$
$11 - 4 = 7$

Second differences

$3 - 1 = 2$
$5 - 3 = 2$
$7 - 5 = 2$

3. A ***recursive function*** is one in which each function value is based on a previous value (or values) of the function.

4. When building a model function, functions can be added, subtracted, or multiplied together. The result will still be a function. This includes linear, quadratic, exponential, and constant functions.

REVIEW EXAMPLES

1) Annie is framing a photo with a length of 6 inches and a width of 4 inches. The distance from the edge of the photo to the edge of the frame is x inches. The combined area of the photo and frame is 63 square inches.

   a. Write a quadratic function to find the distance from the edge of the photo to the edge of the frame.

   b. How wide is the photo and frame together?

Solution:

a. The length of the photo and frame is $x + 6 + x = 6 + 2x$. The width of the photo and frame is $x + 4 + x = 4 + 2x$. The area of the frame is $(6 + 2x)(4 + 2x) = 4x^2 + 20x + 24$. Set this expression equal to the area: $63 = 4x^2 + 20x + 24$.

b. Solve the equation for x.

$$63 = 4x^2 + 20x + 24$$
$$0 = 4x^2 + 20x - 39$$
$$x = -6.5 \text{ or } x = 1.5$$

Length cannot be negative, so the distance from the edge of the photo to the edge of the frame is 1.5 inches. Therefore, the width of the photo and frame together is $4 + 2x = 4 + 2(1.5) = 7$ inches.

2) A scuba diving company currently charges \$100 per dive. On average, there are 30 customers per day. The company performed a study and learned that for every \$20 price increase, the average number of customers per day would be reduced by 2.

   a. The total revenue from the dives is the price per dive multiplied by the number of customers. What is the revenue after 4 price increases?

   b. Write a quadratic equation to represent x price increases.

   c. What price would give the greatest revenue?

Solution:

a. Make a table to show the revenue after 4 price increases.

Number of Price Increases	**Price per Dive (\$)**	**Number of Customers per Day**	**Revenue per Day (\$)**
0	100	30	(100)(30) = 3,000
1	100 + 20(1) = 120	30 – 2(1) = 28	(120)(28) = 3,360
2	100 + 20(2) = 140	30 – 2(2) = 26	(140)(26) = 3,640
3	100 + 20(3) = 160	30 – 2(3) = 24	(160)(24) = 3,840
4	100 + 20(4) = 180	30 – 2(4) = 22	(180)(22) = 3,960

The revenue after 4 price increases is (180)(22) = \$3,960.

b. The table shows a pattern. The price per dive for x price increases is $100 + 20x$. The number of customers for x price increases is $30 - 2x$. So, the equation $y = (100 + 20x)(30 - 2x) = -40x^2 + 400x + 3{,}000$ represents the revenue for x price increases.

c. To find the price that gives the greatest revenue, first find the number of price increases that gives the greatest value. This occurs at the vertex.

Use $\frac{-b}{2a}$ with $a = -40$ and $b = 400$.

$$\frac{-b}{2a} = \frac{-400}{2(-40)} = \frac{-400}{-80} = 5$$

The maximum revenue occurs after 5 price increases.

$100 + 20(5) = 200$

The price of \$200 per dive gives the greatest revenue.

3) Consider the sequence 2, 6, 12, 20, 30, …

a. What explicit expression can be used to find the next term in the sequence?

b. What is the tenth term of the sequence?

Solution:

a. The difference between terms is not constant so the operation involves multiplication. Make a table to try to determine the relationship between the number of the term and the value of the term.

Term number	Term value	Relationship
1	2	1•2
2	6	2•3
3	12	3•4
4	20	4•5
5	30	5•6

Notice the pattern: The value of each term is the product of the term number and one more than the term number. So, the expression is $n(n + 1)$ or $n^2 + n$.

b. The tenth term is $n^2 + n = (10)^2 + (10) = 110$.

EOCT Practice Items

1) What explicit expression can be used to find the next term in this sequence?

2, 8, 18, 32, 50, …

A. $2n$

B. $2n + 6$

C. $2n^2$

D. $2n^2 + 1$

[Key: C]

2) The function $s(t) = vt + h – 0.5at^2$ represents the height of an object, *s*, from the ground after time, *t*, when the object is thrown with an initial velocity of *v*, at an initial height of *h*, and where *a* is the acceleration due to gravity (32 feet per second squared).

A baseball player hits a baseball 4 feet above the ground with an initial velocity of 80 feet per second. About how long will it take the baseball to hit the ground?

A. 2 seconds

B. 3 seconds

C. 4 seconds

D. 5 seconds

[Key: D]

3) A café's annual income depends on *x*, the number of customers. The function $I(x) = 4x^2 - 20x$ describes the café's total annual income. The function $C(x) = 2x^2 + 5$ describes the total amount the café spends in a year. The café's annual profit, *P*(*x*), is the difference between the annual income and the amount spent in a year.

Which function describes *P*(*x*)?

A. $P(x) = 2x^2 - 20x - 5$

B. $P(x) = 4x^3 - 20x^2$

C. $P(x) = 6x^2 - 20x + 5$

D. $P(x) = 8x^4 - 40x^3 - 20x^2 - 100x$

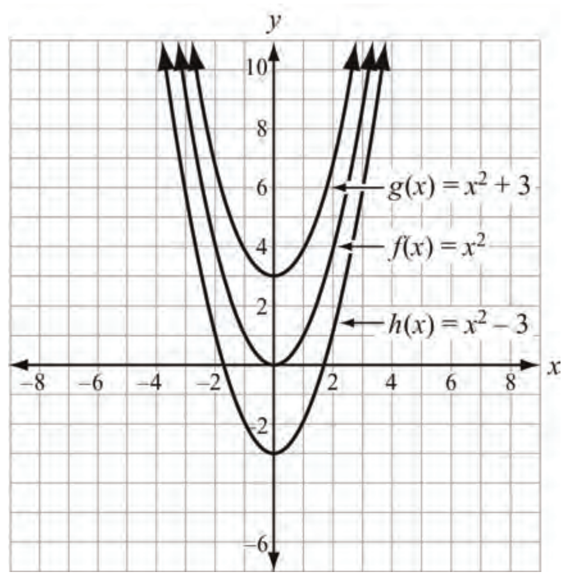
[Key: A]

BUILD NEW FUNCTIONS FROM EXISTING FUNCTIONS

KEY IDEAS

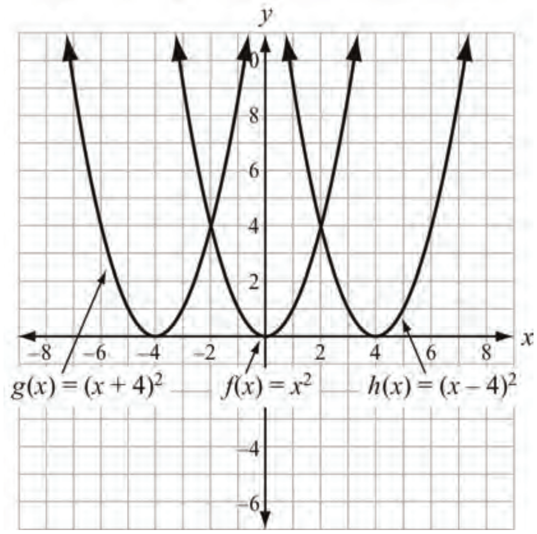
1. A ***parent function*** is the basic function from which all the other functions in a function family are modeled. For the quadratic function family, the parent function is $f(x) = x^2$.

2. For a parent function $f(x)$ and a real number k:
   - The function $f(x) + k$ will move the graph of $f(x)$ up by k units.
   - The function $f(x) – k$ will move the graph of $f(x)$ down by k units.



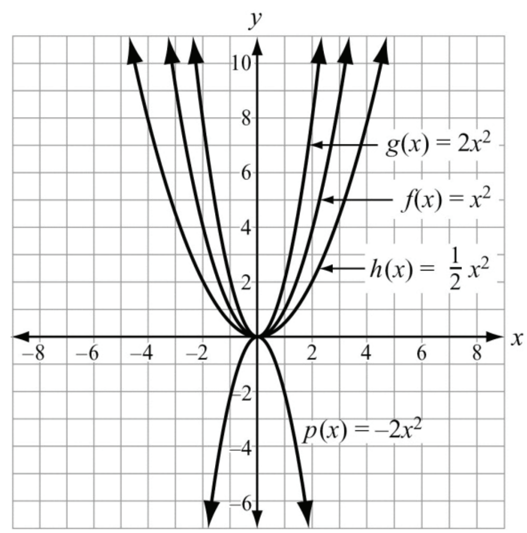
3. For a parent function $f(x)$ and a real number k:

- The function $f(x + k)$ will move the graph of $f(x)$ left by k units.
- The function $f(x - k)$ will move the graph of $f(x)$ right by k units.



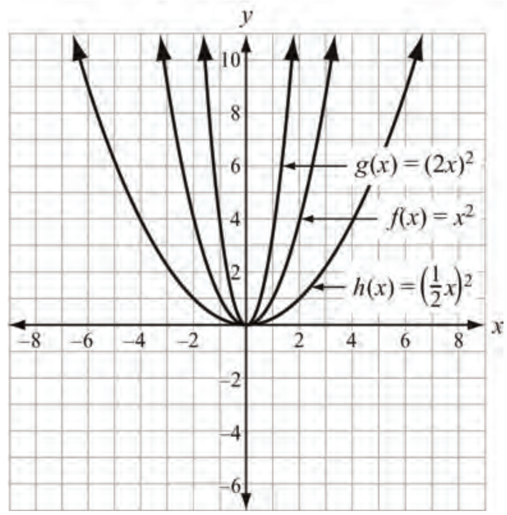
4. For a parent function $f(x)$ and a real number k:

- The function $kf(x)$ will vertically stretch the graph of $f(x)$ by a factor of k units for $|k| > 1$.
- The function $kf(x)$ will vertically shrink the graph of $f(x)$ by a factor of k units for $|k| < 1$.
- The function $kf(x)$ will reflect the graph of $f(x)$ over the x-axis for negative values of k.



5. For a parent function $f(x)$ and a real number k:

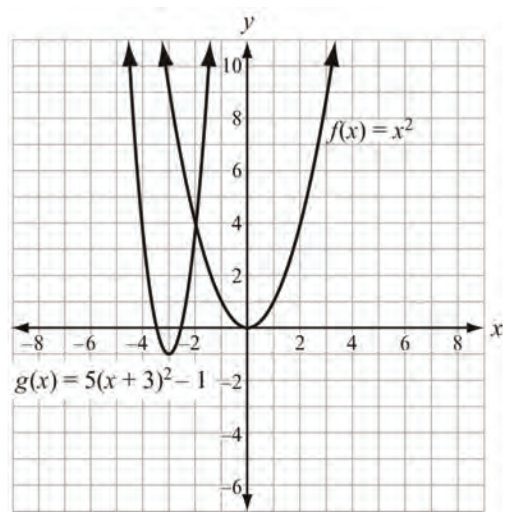
- The function $f(kx)$ will horizontally shrink the graph of $f(x)$ by a factor of $\frac{1}{k}$ units for $|k| > 1$.
- The function $f(kx)$ will horizontally stretch the graph of $f(x)$ by a factor of $\frac{1}{k}$ units for $|k| < 1$.
- The function $f(kx)$ will reflect the graph of $f(x)$ over the y-axis for negative values of k.



6. You can apply more than one of these changes at a time to a parent function.

Example:

$f(x) = 5(x + 3)^2 – 1$ will translate $f(x) = x^2$ left 3 units and down 1 unit and stretch the function vertically by a factor of 5.



7. Functions can be classified as even or odd.

- If a graph is symmetric to the y-axis, then it is an ***even function***.
  That is, if $f(-x) = f(x)$, then the function is even.

- If a graph is symmetric to the origin, then it is an ***odd function***.
  That is, if $f(-x) = -f(x)$, then the function is odd.

Important Tip

Remember that when you change $f(x)$ to $f(x+k)$, move the graph to the **left** when k is positive, and to the **right** when k is negative. This may seem different from what you would expect, so be sure to understand why this occurs in order to apply the shifts correctly.

REVIEW EXAMPLES

1) Compare the graphs of the following functions to $f(x)$.

   a. $\frac{1}{2}f(x)$
   b. $f(x) - 5$
   c. $f(x - 2) + 1$

Solution:

   a. The graph of $\frac{1}{2}f(x)$ is a vertical shrink of $f(x)$ by a factor of $\frac{1}{2}$.
   b. The graph of $f(x) - 5$ is a shift of the graph of $f(x)$ down 5 units.
   c. The graph of $f(x - 2) + 1$ is a shift of the graph of $f(x)$ right 2 units and up 1 unit.

2) Is $f(x) = 2x^3 + 6x$ even, odd, or neither? Explain how you know.

Solution:

Substitute $-x$ for x and evaluate:

$$\begin{aligned} f(-x) &= 2(-x)^3 + 6(-x) \\ &= -2x^3 - 6x \\ &= -(2x^3 + 6x) \end{aligned}$$

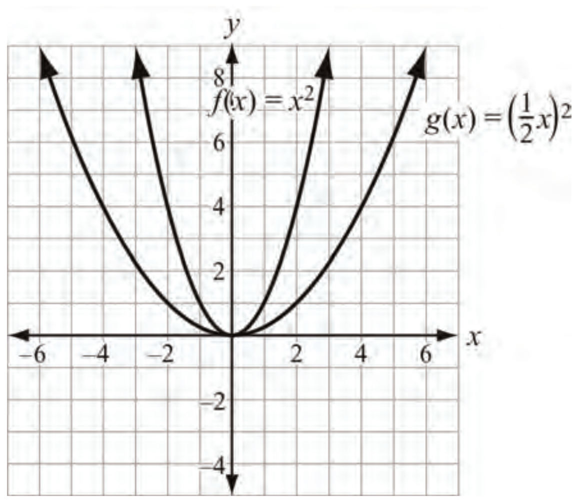
$f(-x)$ is the opposite of $f(x)$, so the function is odd.

3) How does the graph of $f(x)$ compare to the graph of $f\left(\frac{1}{2}x\right)$?

Solution:

The graph of $f\left(\frac{1}{2}x\right)$ is a horizontal stretch of $f(x)$ by a factor of 2. The graphs of $f(x)$ and $g(x) = f\left(\frac{1}{2}x\right)$ are shown.

For example, at $y = 4$, the width of $f(x)$ is 4 and the width of $g(x)$ is 8. So, the graph of $g(x)$ is wider than $f(x)$ by a factor of 2.



EOCT Practice Items

1) Which statement BEST describes the graph of $f(x + 6)$?

A. The graph of $f(x)$ is shifted up 6 units.

B. The graph of $f(x)$ is shifted left 6 units.

C. The graph of $f(x)$ is shifted right 6 units.

D. The graph of $f(x)$ is shifted down 6 units.

[Key: B]

2) Which of these is an even function?

A. $f(x) = 5x^2 – x$

B. $f(x) = 3x^3 + x$

C. $f(x) = 6x^2 – 8$

D. $f(x) = 4x^3 + 2x^2$

[Key: C]

3) Which statement BEST describes how the graph of $g(x) = –3x^2$ compares to the graph of $f(x) = x^2$?

A. The graph of $g(x)$ is a vertical stretch of $f(x)$ by a factor of 3.

B. The graph of $g(x)$ is a reflection of $f(x)$ across the x-axis.

C. The graph of $g(x)$ is a vertical shrink of $f(x)$ by a factor of $\frac{1}{3}$ and a reflection across the x-axis.

D. The graph of $g(x)$ is a vertical stretch of $f(x)$ by a factor of 3 and a reflection across the x-axis.

[Key: D]

CONSTRUCT AND COMPARE LINEAR, QUADRATIC, AND EXPONENTIAL MODELS TO SOLVE PROBLEMS

KEY IDEAS

1. ***Exponential functions*** have a fixed number as the base and a variable number as the exponent.

2. The value of an exponential function with a base greater than 1 will eventually exceed the value of a quadratic function. Similarly, the value of a quadratic function will eventually exceed the value of a linear function.

Example:

Exponential	
x	$y = 2^x$
1	2
2	4
3	8
4	16
5	32
6	64

Quadratic	
x	$y = x^2 + 2$
1	3
2	6
3	11
4	18
5	27
6	38

Linear	
x	$y = x + 2$
1	3
2	4
3	5
4	6
5	7
6	8

REVIEW EXAMPLES

1) This table shows that the value of $f(x) = 5x^2 + 4$ is greater than the value of $g(x) = 2^x$ over the interval [0, 8].

x	$f(x)$	$g(x)$
0	$5(0)^2 + 4 = 4$	$2^0 = 1$
2	$5(2)^2 + 4 = 24$	$2^2 = 4$
4	$5(4)^2 + 4 = 84$	$2^4 = 16$
6	$5(6)^2 + 4 = 184$	$2^6 = 64$
8	$5(8)^2 + 4 = 324$	$2^8 = 256$

As x increases, will the value of $f(x)$ always be greater than the value of $g(x)$? Explain how you know.

Solution:

For some value of x, the value of an exponential function will eventually exceed the value of a quadratic function. To demonstrate this, find the values of $f(x)$ and $g(x)$ for another value of x, such as $x = 10$.

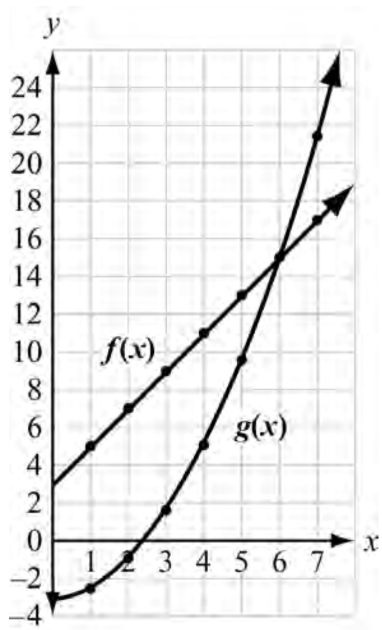
$f(x) = 5(10)^2 + 4 = 504$
$g(x) = 2^{10} = 1{,}024$

In fact, this means that for some value of x between 8 and 10, the value of $g(x)$ becomes greater than the value of $f(x)$ and remains greater for all subsequent values of x.

2) How does the growth rate of the function $f(x) = 2x + 3$ compare with $g(x) = 0.5x^2 – 3$? Use a graph to explain your answer.

Solution:

Graph $f(x)$ and $g(x)$ over the interval $x \geq 0$.



The graph of $f(x)$ increases at a constant rate because it is linear.

The graph of $g(x)$ increases at an increasing rate because it is quadratic.

The graphs can be shown to intersect at (6, 15), and the value of $g(x)$ is greater than the value of $f(x)$ for $x > 6$.

EOCT Practice Items

1) A table of values is shown for $f(x)$ and $g(x)$.

x	$f(x)$
0	0
1	1
2	4
3	9
4	16
5	25

x	$g(x)$
0	–2
1	–1
2	1
3	5
4	13
5	29

Which statement compares the graphs of $f(x)$ and $g(x)$ over the interval [0, 5]?

A. The graph of $f(x)$ always exceeds the graph of $g(x)$ over the interval [0, 5].

B. The graph of $g(x)$ always exceeds the graph of $f(x)$ over the interval [0, 5].

C. The graph of $g(x)$ exceeds the graph of $f(x)$ over the interval [0, 4], the graphs intersect at a point between 4 and 5, and then the graph of $f(x)$ exceeds the graph of $g(x)$.

D. The graph of $f(x)$ exceeds the graph of $g(x)$ over the interval [0, 4], the graphs intersect at a point between 4 and 5, and then the graph of $g(x)$ exceeds the graph of $f(x)$.

[Key: D]

2) Which statement is true about the graphs of exponential functions?

A. The graphs of exponential functions never exceed the graphs of linear and quadratic functions.

B. The graphs of exponential functions always exceed the graphs of linear and quadratic functions.

C. The graphs of exponential functions eventually exceed the graphs of linear and quadratic functions.

D. The graphs of exponential functions eventually exceed the graphs of linear functions, but not quadratic functions.

[Key: C]

3) Which statement BEST describes the comparison of the function values for $f(x)$ and $g(x)$?

x	$f(x)$	$g(x)$
0	0	–10
1	2	–9
2	4	–6
3	6	–1
4	8	6

A. The values of $f(x)$ will always exceed the values of $g(x)$.

B. The values of $g(x)$ will always exceed the values of $f(x)$.

C. The values of $f(x)$ exceed the values of $g(x)$ over the interval [0, 5].

D. The values of $g(x)$ begin to exceed the values of $f(x)$ within the interval [4, 5].

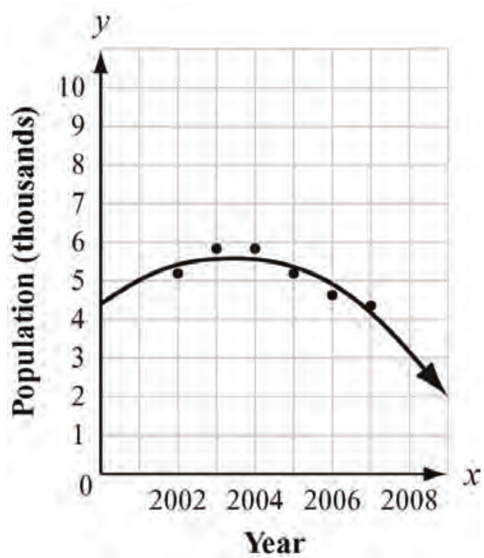
[Key D]

SUMMARIZE, REPRESENT, AND INTERPRET DATA ON TWO CATEGORICAL AND QUANTITATIVE VARIABLES

KEY IDEAS

1. A ***quadratic regression*** equation is a curve of best fit for data given in a scatter plot. The curve most likely will not go through all of the data points, but should come close to most of them.

Example:



2. A quadratic regression equation can be used to make predictions about data. To do this, evaluate the function for a given input value.

REVIEW EXAMPLES

1) Amery recorded the distance and height of a basketball when shooting a free throw.

Distance (feet), *x*	**Height (feet), *f*(*x*)**
0	4
2	8.4
6	12.1
9	14.2
12	13.2
13	10.5
15	9.8

The height of the basketball after x seconds can be approximated by the quadratic function $f(x) = -0.118x^2 + 2.112x + 4.215$. Using this function, what is the approximate maximum height of the basketball?

Solution:

Find the vertex of the function.

$$\frac{-b}{2a} = \frac{-(2.112)}{2(-0.118)} \approx 8.949$$

Substitute 8.949 for x in the original function:

$f(8.949) = -0.118(8.949)^2 + 2.112(8.949) + 4.215 \approx 13.665 \approx 13.7$

The maximum height of the basketball predicted by the function is about 13.7 feet.

2) This table shows the population of a city every ten years since 1970.

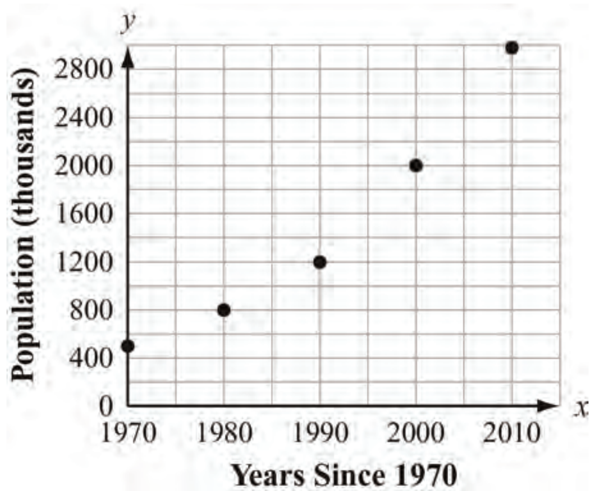
Years Since 1970, *x*	**Population (in thousands), *y***
0	489
10	801
20	1,202
30	1,998
40	2,959

a. Make a scatter plot showing the data.

b. Which type of function better models the relationship between 1970 and 2010, quadratic or linear?

Solution:

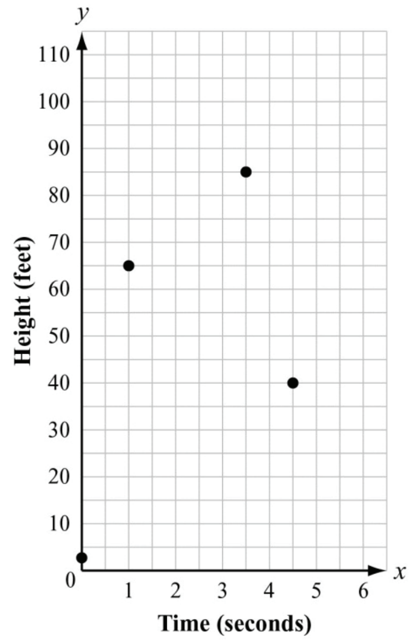
a. Plot the points on a coordinate grid.



b. A quadratic model represents the population better than a linear model.

EOCT Practice Items

1) This scatter plot shows the height, in feet, of a ball launched in the air from an initial height of 3 feet, and the time the ball traveled in seconds.



Based on an estimated quadratic regression curve, which is the BEST estimate for the maximum height of the ball?

A. 75 feet

B. 85 feet

C. 100 feet

D. 120 feet

[Key: C]

2) The quadratic function $f(x) = -45x^2 + 350x + 1{,}590$ models the population of a city, where x is the number of years after 2005 and $f(x)$ is the population of the city in thousands of people. What is the estimated population of the city in 2015?

A. 45,000

B. 77,000

C. 590,000

D. 670,000

[Key: C]

Unit 6: Modeling Geometry

This unit investigates coordinate geometry. Students look at equations for circles and parabolas and use given information to derive equations for representations of these figures on a coordinate plane. Students also use coordinates to prove simple geometric theorems using the properties of distance, slope, and midpoints. Students will verify whether a figure is a special quadrilateral by showing that sides of figures are parallel or perpendicular.

KEY STANDARDS

Solve systems of equations

MCC9-12.A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.

Translate between the geometric description and the equation for a conic section

MCC9-12.G.GPE.1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

MCC9-12.G.GPE.2 Derive the equation of a parabola given a focus and directrix.

Use coordinates to prove simple geometric theorems algebraically

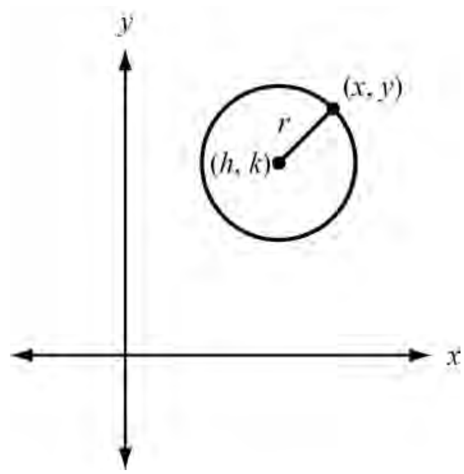
MCC9-12.G.GPE.4 Use coordinates to prove simple geometric theorems algebraically. *For example, prove or disprove that a figure defined by four given points in the coordinate plane is a rectangle; prove or disprove that the point (1, $\sqrt{3}$) lies on the circle centered at the origin and containing the point (0, 2).* (*Restrict to context of circles and parabolas.*)

TRANSLATE BETWEEN THE GEOMETRIC DESCRIPTION AND THE EQUATION FOR A CONIC SECTION

KEY IDEAS

1. A ***circle*** is the set of points in a plane equidistant from a given point, or center, of the circle.

2. The standard form of the equation of a circle is $(x-h)^2+(y-k)^2=r^2$, where (h, k) is the center of the circle and r is the radius of the circle.



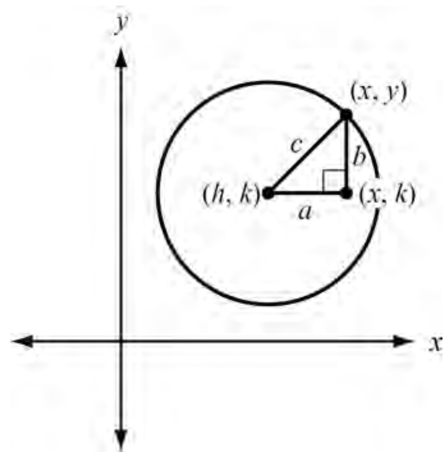
3. The equation of a circle can be derived from the Pythagorean Theorem.

Example:

Given a circle with a center at (h, k) and a point (x, y) on the circle, draw a horizontal line segment from (h, k) to (x, k). Label this line segment a. Draw a vertical line segment from (x, y) to (x, k). Label this line segment b. Label the radius c. A right triangle is formed.

The length of line segment a is given by $(x - h)$.

The length of line segment b is given by $(y - k)$.

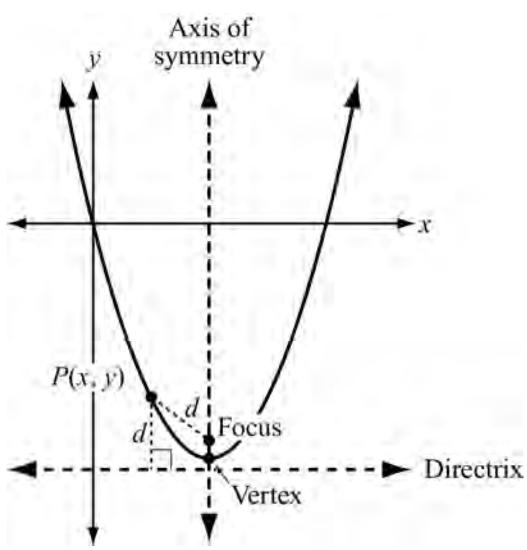


Using the Pythagorean Theorem, substitute $(x - h)$ for a, $(y - k)$ for b, and r for c in the equation.

$a^2 + b^2 = c^2$	Use the Pythagorean Theorem.
$(x - h)^2 + (y - k)^2 = r^2$	Substitution.

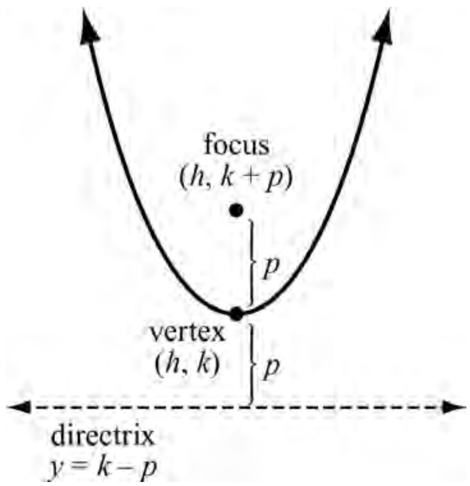
The equation for a circle with a center at (h, k) and a radius r is: $(x-h)^2+(y-k)^2=r^2$.

4. A ***parabola*** is the set of all points equidistant from a given point, the ***focus*** of the parabola, and a given line, the ***directrix***.

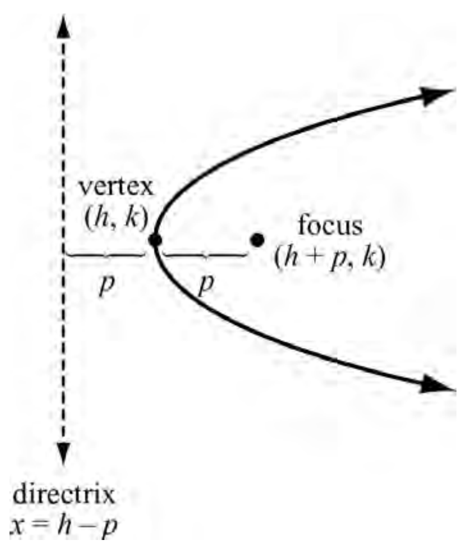


5. The form of the equation of a parabola depends on whether the directrix of the parabola is horizontal or vertical.

When the directrix of the parabola is horizontal, the equation of the parabola is $y-k=\frac{1}{4p}(x-h)^2$, where (h, k) is the vertex, $(h, k+p)$ is the focus, $x = h$ is the ***axis of symmetry***, and $y = k - p$ is the equation of the directrix. If $p > 0$, the parabola opens up, and if $p < 0$, the parabola opens down.

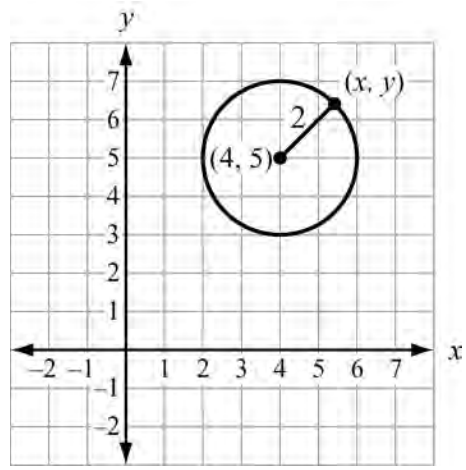


When the directrix of the parabola is vertical, the equation of the parabola is $x-h=\frac{1}{4p}(y-k)^2$, where (h, k) is the vertex, $(h+p, k)$ is the focus, $y = k$ is the axis of symmetry, and $x = h - p$ is the equation of the directrix. If $p > 0$, the parabola opens to the right, and if $p < 0$, the parabola opens to the left.



REVIEW EXAMPLES

1) What is the equation of the circle with a center at (4, 5) and a radius of 2?



Solution:

Use the standard form for the equation for a circle, $(x-h)^2+(y-k)^2=r^2$. Substitute the values into the equation, with $h = 4$, $k = 5$, and $r = 2$.

$(x-h)^2+(y-k)^2=r^2$	Equation for a circle.
$(x-(4))^2+(y-(5))^2=(2)^2$	Substitute the values in the equation of a circle.
$(x-4)^2+(y-5)^2=4$	Evaluate.

The equation of a circle with a center at (4, 5) and a radius of 2 is $(x-4)^2+(y-5)^2=4$.

2) What is the center and radius of the circle given by $8x^2+8y^2-16x-32y+24=0$?

Solution:

Write the equation in standard form to identify the center and radius of the circle. First, write the equation so the x-terms are next to each other and the y-terms are next to each other, both on the left side of the equation, and the constant term is on the right side of the equation.

$8x^2+8y^2-16x-32y+24=0$	Original equation.
$8x^2+8y^2-16x-32y=-24$	Subtract 24 from both sides.
$8x^2-16x+8y^2-32y=-24$	Commutative Property of Addition.
$x^2-2x+y^2-4y=-3$	Divide both sides by 8.
$(x^2-2x)+(y^2-4y)=-3$	Associative Property of Addition.

Next, to write the equation in standard form, complete the square for the x-terms and the y-terms. Find $\left(\frac{b}{2a}\right)^2$ for the x- and y-terms.

$$x\text{-term: } \left(\frac{b}{2a}\right)^2 = \left(\frac{-2}{2(1)}\right)^2 = (-1)^2 = 1$$

$$y\text{-term: } \left(\frac{b}{2a}\right)^2 = \left(\frac{-4}{2(1)}\right)^2 = (-2)^2 = 4$$

$(x^2 - 2x + 1) + (y^2 - 4y + 4) = -3 + 1 + 4$	Add 1 and 4 to each side of the equation.
$(x - 1)^2 + (y - 2)^2 = 2$	Write the trinomials as squares of binomials.

This equation for the circle is written in standard form, where $h = 1$, $k = 2$, and $r^2 = 2$. The center of the circle is (1, 2) and the radius is $\sqrt{2}$.

3) Write an equation for the parabola with a focus at (6, 4) and a directrix of $y = 2$.

Solution:

The standard form of an equation of a parabola with a horizontal directrix is:
$y - k = \frac{1}{4p}(x - h)^2$, where (h, k) is the vertex, $(h, k + p)$ is the focus, $x = h$ is the axis of symmetry, and $y = k - p$ is the equation of the directrix.
Because the directrix is below the focus, the parabola opens upward.
Based on the x-coordinate of the focus, $h = 6$.
Based on the y-coordinate of the focus, $k + p = 4$.
Based on the equation for the directrix, $k - p = 2$.
Solve a system of equations to find k and p.

$$\begin{cases} k + p = 4 \\ k - p = 2 \end{cases}$$

$2k = 6$	Add the equations.
$k = 3$	Division Property of Equality.
$3 + p = 4$	Substitute the value of k in the equation to find p.
$p = 1$	Subtraction Property of Equality.

Substitute 6 for h, 3 for k, and 1 for p in the standard form of a parabola, $y - k = \frac{1}{4p}(x - h)^2$.

An equation for the parabola with a focus at (6, 4) and a directrix of $y = 2$ is $y - 3 = \frac{1}{4}(x - 6)^2$.

EOCT Practice Items

1) Which is an equation for the circle with a center at (–2, 3) and a radius of 3?

A. $x^2 + y^2 + 4x - 6y + 22 = 0$

B. $2x^2 + 2y^2 + 3x - 3y + 4 = 0$

C. $x^2 + y^2 + 4x - 6y + 4 = 0$

D. $3x^2 + 3y^2 + 4x - 6y + 4 = 0$

[Key: C]

2) What is the center of the circle given by the equation $x^2 + y^2 - 10x - 11 = 0$?

A. (5, 0)

B. (0, 5)

C. (–5, 0)

D. (0, –5)

[Key: A]

3) Which shows an equation for the parabola with a focus at (10, 0) and a directrix of $x = 2$?

A. $x = \frac{1}{4}(y-6)^2$

B. $x = \frac{1}{16}(y-6)^2$

C. $x - 6 = \frac{1}{4}y^2$

D. $x - 6 = \frac{1}{16}y^2$

[Key: D]

4) Which shows an equation for the parabola with a focus at (4, –5) and a directrix of *y* = –1?

A. $y+3=\frac{1}{4}(x-4)^2$

B. $y-4=\frac{1}{4}(x+3)^2$

C. $y+3=-\frac{1}{8}(x-4)^2$

D. $y-4=-\frac{1}{8}(x+3)^2$

[Key: C]

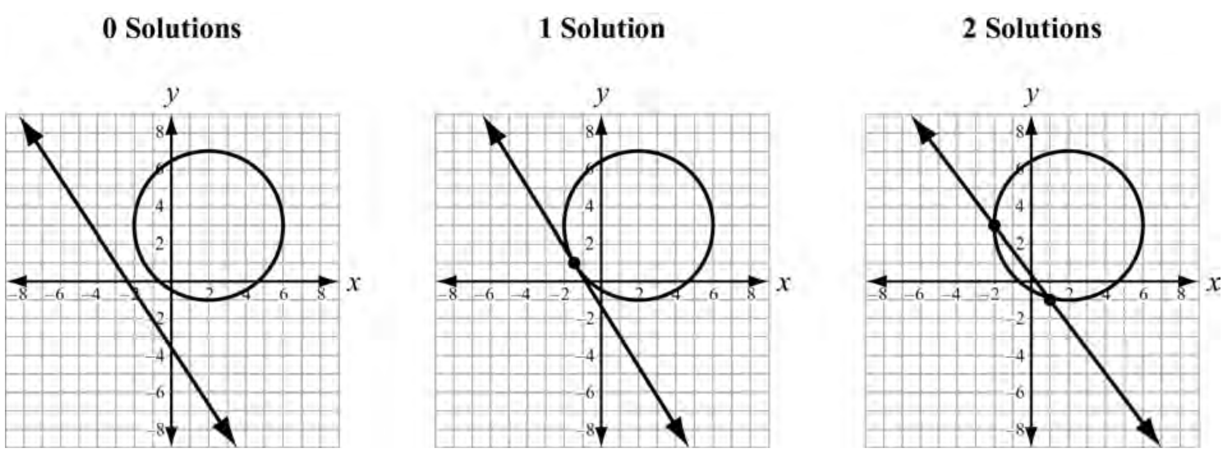
SOLVE SYSTEMS OF EQUATIONS THAT INCLUDE CIRCLES

KEY IDEAS

1. To algebraically solve a system of equations involving a linear equation and an equation representing a circle, first solve the linear equation for a variable. Then substitute into the equation that represents the circle. Once you have found the solution for one variable, substitute the value into either equation and solve for the second variable.

2. To graphically solve a system of equations involving a linear equation and an equation that represents a circle, graph both equations. The point (or points) of intersection are the solutions. Check these solutions by substituting them into both equations.

3. There can be 0, 1, or 2 solutions to a system of equations involving a linear equation and an equation representing a circle. The number of solutions can be seen graphically by the number of points of intersection. When a system of equations involving a linear equation and an equation representing a circle has 0 solutions, the line does not intersect the circle; for 1 solution, the line is tangent to the circle and intersects it at a single point; for 2 solutions, the line is secant to the circle and intersects it at two points.



REVIEW EXAMPLES

1) A circle is centered at the origin and has a radius of $\sqrt{5}$ units. A line with a slope of 2 passes through the origin and intersects the circle in two places. Where does the line intersect the circle?

Solution:

First write an equation for the circle: $x^2 + y^2 = 5$.
Then write an equation for the line: $y = 2x$.
Solve the system of equations algebraically.

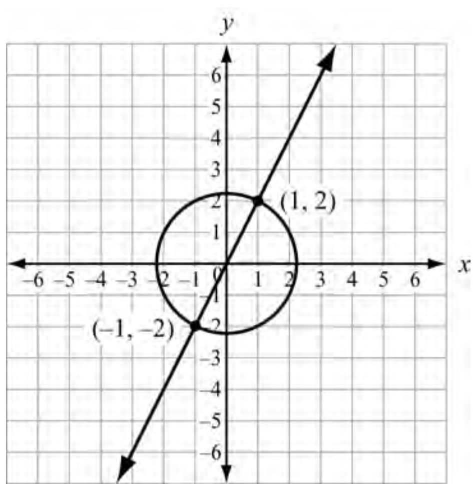
$$\begin{aligned} x^2 + y^2 &= 5 \\ y &= 2x \end{aligned} \qquad \begin{aligned} x^2 + (2x)^2 &= 5 \\ x^2 + 4x^2 &= 5 \\ 5x^2 &= 5 \\ x^2 &= 1 \\ x &= \pm 1 \end{aligned}$$

To find the *y*-coordinate for the solutions, you can substitute the values for *x* into either equation. Substitute 1 and –1 into the linear equation and solve for *y*.

$$\begin{aligned} y &= 2x \\ y &= 2(1) \\ y &= 2 \end{aligned} \qquad \begin{aligned} y &= 2x \\ y &= 2(-1) \\ y &= -2 \end{aligned}$$

The solutions are (1, 2) and (–1, –2), so the line intersects the circle at (1, 2) and (–1, –2).

For the system of equations given above, you can also graph the equations on a coordinate plane. The line appears to intersect the circle at (–1, –2) and (1, 2). Check these solutions by substituting them into both equations.



2) Given the system of equations:

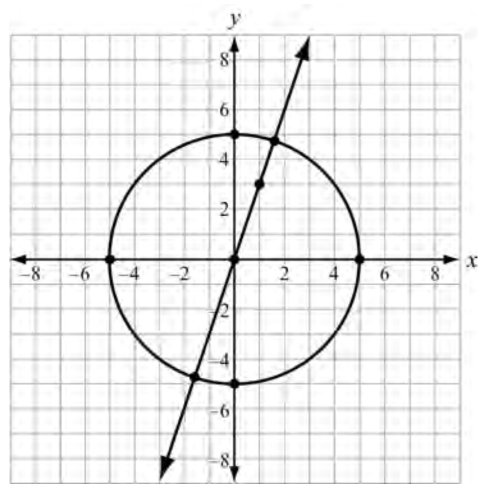
$$x^2 + y^2 = 25$$
$$y = 3x$$

a. Sketch a graph of the system to determine the number of solutions.

b. Solve the system algebraically to find the solutions.

Solution:

a. The first equation is a circle with a radius of 5 and a center at (0, 0). To sketch the circle, plot points 5 units from the center: (5, 0), (–5, 0), (0, 5), and (0, –5). Connect the points to form a circle. The second equation is a line with a slope of 3 that goes through point (0, 0). To sketch the line, plot the point (0, 0). Use the slope 3 to identify another point, (1, 3), and connect the points to form a line.



The line intersects the circle at two points, so there are 2 solutions.

b. Solve the system algebraically.

$$x^2 + y^2 = 25$$
$$y = 3x$$

Substitute $3x$ from the second equation for y in the first equation. Then solve for x.

$x^2 + (3x)^2 = 25$	Substitute $3x$ for y.
$x^2 + 9x^2 = 25$	Power of a product.
$10x^2 = 25$	Add.
$x^2 = 2.5$	Divide both sides of the equation by 10.
$x = \pm\sqrt{2.5}$	Take the square root of both sides of the equation.
$x \approx \pm 1.58$	Evaluate and round.

To find the *y*-coordinates of the solutions, substitute the values for $x = -\sqrt{2.5}$ and $x = \sqrt{2.5}$ into either equation and solve for *y*.

$$\begin{array}{ll} y = 3x & y = 3x \\ y = -3\sqrt{2.5} & y = 3\sqrt{2.5} \\ y \approx -4.74 & y \approx 4.74 \end{array}$$

The solutions are approximately (–1.58, –4.74) and (1.58, 4.74).

EOCT Practice Items

1) A circle is centered at the origin and has a radius of 3 units. A horizontal line passes through the point (0, 3). In how many places does the line intersect the circle?

A. 0

B. 1

C. 2

D. infinitely many

[Key: B]

2) A circle is centered at the origin and has a radius of $\sqrt{10}$ units. A line has a slope of –3 and passes through the origin. At which points does the line intersect the circle?

A. (–3, 1) and (3, –1)

B. (–1, 3) and (1, –3)

C. (1, 3) and (–1, –3)

D. (3, 1) and (–3, –1)

[Key: B]

USE COORDINATES TO PROVE SIMPLE GEOMETRIC THEOREMS ALGEBRAICALLY

KEY IDEAS

1. Given the equation of a circle or a parabola, you can verify whether a point lies on the circle or on the parabola by substituting the coordinates of the point into the equation. If the resulting equation is true, then the point lies on the figure. If the resulting equation is not true, then the point does not lie on the figure.

2. Given the center and radius of a circle, you can verify whether a point lies on the circle by determining whether the distance between the given point and the center is equal to the radius.

3. Given the focus and directrix of a parabola, you can verify whether a point lies on the parabola by determining whether the distance between the given point and the directrix is equal to the distance between the given point and the focus.

4. To prove properties about special parallelograms on a coordinate plane, you can use the midpoint, distance, and slope formulas:

   - The ***midpoint formula*** is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. This formula is used to find the coordinates of the midpoint of $\overline{AB}$, given $A(x_1, y_1)$ and $B(x_2, y_2)$.
   - The ***distance formula*** is $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. This formula is used to find the length of $\overline{AB}$, given $A(x_1, y_1)$ and $B(x_2, y_2)$.
   - The ***slope formula*** is $m=\left(\frac{y_2-y_1}{x_2-x_1}\right)$. This formula is used to find the slope of a line or line segment, given any two points on the line or line segment $A(x_1, y_1)$ and $B(x_2, y_2)$.

5. To prove a triangle is isosceles, you can use the distance formula to show that at least two sides are congruent.

6. You can use properties of quadrilaterals to help prove theorems:

- To prove a quadrilateral is a parallelogram, show that the opposite sides are parallel using slope.
- To prove a quadrilateral is a rectangle, show that the opposite sides are parallel and the consecutive sides are perpendicular using slope.
- To prove a quadrilateral is a rhombus, show that all four sides are congruent using distance formula.
- To prove a quadrilateral is a square, show that all four sides are congruent and consecutive sides are perpendicular using slope and distance formula.

7. You can also use diagonals of a quadrilateral to help prove theorems:

- To prove a quadrilateral is a parallelogram, show that its diagonals bisect each other.
- To prove a quadrilateral is a rectangle, show that its diagonals bisect each other and are congruent.
- To prove a quadrilateral is a rhombus, show that its diagonals bisect each other and are perpendicular.
- To prove a quadrilateral is a square, show that its diagonals bisect each other, are congruent, and are perpendicular.

Important Tips

- When using the formulas for midpoint, distance, and slope, the order of the points does not matter. You can use either point to be (x_1, y_1) and (x_2, y_2), but be careful to always subtract in the same order.
- Parallel lines have the same slope. Perpendicular lines have slopes that are the negative reciprocal of each other.

REVIEW EXAMPLES

1) Quadrilateral *ABCD* has vertices *A*(–1, 3), *B*(3, 5), *C*(4, 3), and *D*(0, 1). Is *ABCD* a rectangle? Explain how you know.

Solution:

First determine whether or not the figure is a parallelogram. If the figure is a parallelogram, then the diagonals bisect each other. If the diagonals bisect each other, then the midpoints of the diagonals are the same point. Use the midpoint formula to determine the midpoints for each diagonal.

$$\text{Midpoint } \overline{AC}: \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-1+4}{2}, \frac{3+3}{2}\right) = \left(\frac{3}{2}, \frac{6}{2}\right) = (1.5,\ 3)$$
$$\text{Midpoint } \overline{BD}: \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{3+0}{2}, \frac{5+1}{2}\right) = \left(\frac{3}{2}, \frac{6}{2}\right) = (1.5,\ 3)$$

The diagonals have the same midpoint; therefore, the diagonals bisect each other and the figure is a parallelogram.

A parallelogram with congruent diagonals is a rectangle. Determine whether or not the diagonals are congruent.

Use the distance formula to find the length of the diagonals:

$$AC = \sqrt{(4-(-1))^2 + (3-3)^2} = \sqrt{(5)^2 + (0)^2} = \sqrt{25+0} = \sqrt{25} = 5$$
$$BD = \sqrt{(0-3)^2 + (1-5)^2} = \sqrt{(-3)^2 + (-4)^2} = \sqrt{9+16} = \sqrt{25} = 5$$

The diagonals are congruent because they have the same length.

The figure is a parallelogram with congruent diagonals, so the figure is a rectangle.

2) Circle *C* has a center of (–2, 3) and a radius of 4. Does point (–4, 6) lie on circle *C*?

Solution:

The distance from any point on the circle to the center of the circle is equal to the radius. Use the distance formula to find the distance from (–4, 6) to the center (–2, 3). Then see if it is equal to the radius, 4.

$\sqrt{(-4-(-2))^2+(6-3)^2}$	Substitute the coordinates of the points in the distance formula.
$\sqrt{(-2)^2+(3)^2}$	Evaluate within parentheses.
$\sqrt{4+9}$	Evaluate the exponents.
$\sqrt{13}$	Add.

The distance from (–4, 6) to (–2, 3) is not equal to the radius, so (–4, 6) does not lie on the circle. (In fact, since $\sqrt{13} < 4$, the distance is less than the radius, so the point lies inside of the circle.)

3) A parabola has its focus at (3, 4) and its directrix at $y = -2$. Does the point (–9, 13) lie on the parabola?

Solution:

Each point on the parabola is equidistant from the focus and the directrix, $y = -2$. Because the directrix is a horizontal line, you can find the distance from any point to the directrix using the y-coordinates. The distance from point (–9, 13) to $y = -2$ is $|13 - (-2)| = 15$. If the given point is on the parabola, the distance between the point and the focus must also be 15.

Use the distance formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ to determine the distance from the given point to the focus. Then see if it is equal to the distance between that point and the directrix.

$\sqrt{(-9-3)^2 + (13-4)^2}$	Substitute the coordinates of the points in the distance formula.
$\sqrt{(-12)^2 + (9)^2}$	Evaluate within parentheses.
$\sqrt{144+81}$	Evaluate the exponents.
$\sqrt{225}$	Add.
15	Evaluate the square root.

The point (–9, 13) is the same distance from the focus and the directrix, so point (–9, 13) lies on the parabola.

EOCT Practice Items

1) A parabola has a focus at (–3, 6) and a directrix of y = –4. For which value of a does the point (a, 6) lie on the parabola?

A. 1

B. 7

C. 10

D. 13

[Key: B]

2) Which information is needed to show that a parallelogram is a rectangle?

A. The diagonals bisect each other.

B. The diagonals are congruent.

C. The diagonals are congruent and perpendicular.

D. The diagonals bisect each other and are perpendicular.

[Key: B]

3) Which point is on a circle with a center of (3, –9) and a radius of 5?

A. (–6, 5)

B. (–1, 6)

C. (1, 6)

D. (6, –5)

[Key: D]

Unit 7: Applications of Probability

This unit investigates the concept of probability. Students look at sample spaces and identify unions, intersections, and complements. They identify ways to tell if events are independent. The concept of conditional probability is related to independence and students use the concepts to solve real-world problems, including those that are presented in two-way frequency tables. Students find probabilities of compound events using the rules of probability.

KEY STANDARDS

Understand independence and conditional probability and use them to interpret data

MCC9-12.S.CP.1 Describe events as subsets of a sample space (the set of outcomes) using characteristics (or categories) of the outcomes, or as unions, intersections, or complements of other events ("or," "and," "not").★

MCC9-12.S.CP.2 Understand that two events A and B are independent if the probability of A and B occurring together is the product of their probabilities, and use this characterization to determine if they are independent.★

MCC9-12.S.CP.3 Understand the conditional probability of A given B as P(A and B)/P(B), and interpret independence of A and B as saying that the conditional probability of A given B is the same as the probability of A, and the conditional probability of B given A is the same as the probability of B.★

MCC9-12.S.CP.4 Construct and interpret two-way frequency tables of data when two categories are associated with each object being classified. Use the two-way table as a sample space to decide if events are independent and to approximate conditional probabilities. *For example, collect data from a random sample of students in your school on their favorite subject among math, science, and English. Estimate the probability that a randomly selected student from your school will favor science given that the student is in tenth grade. Do the same for other subjects and compare the results*.★

MCC9-12.S.CP.5 Recognize and explain the concepts of conditional probability and independence in everyday language and everyday situations. *For example, compare the chance of having lung cancer if you are a smoker with the chance of being a smoker if you have lung cancer*.★

Use the rules of probability to compute probabilities of compound events in a uniform probability model

MCC9-12.S.CP.6 Find the conditional probability of A given B as the fraction of B's outcomes that also belong to A, and interpret the answer in terms of the model.★

MCC9-12.S.CP.7 Apply the Addition Rule, P(A or B) = P(A) + P(B) – P(A and B), and interpret the answer in terms of the model.★

UNDERSTAND INDEPENDENCE AND CONDITIONAL PROBABILITY AND USE THEM TO INTERPRET DATA

KEY IDEAS

1. In probability, a ***sample space*** is the set of all possible outcomes. Any subset from the sample space is an ***event***.

2. If the outcome of one event does not rely on the other event, the events are ***independent***. If the outcome of one event relies on the other event, the events are ***dependent***.

3. The ***intersection*** of two or more events is all of the outcomes shared by both events. The intersection is denoted with the word "and," or with the $\cap$ symbol. For example, the intersection of *A* and *B* is shown as $A \cap B$.

4. The ***union*** of two or more events is all of the outcomes for either event. The union is denoted with the word "or," or with the $\cup$ symbol. For example, the union of *A* and *B* is shown as $A \cup B$. The probability of the union of two events that have no outcomes in common is the sum of each individual probability.

5. The ***complement*** of an event is the set of outcomes in the same sample space that are not included in the outcomes of the event. The complement is denoted with the word "not," or with the $'$ symbol. For example, the complement of *A* is shown as A'. The set of outcomes and its complement make up the entire sample space.

6. ***Conditional probabilities*** are found when one event has already occurred and a second event is being analyzed. Conditional probability is denoted $P(A \mid B)$ and is read as "The probability of *A* given *B*."

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)}$$

7. Two events—*A* and *B*—are independent if the probability of the ~~union~~ intersection is the same as the product of each individual probability. That is, ~~$P(A \cup B) = P(A) \cdot P(B)$~~ $P(A \cap B) = P(A) \bullet P(B)$.

8. If two events are independent, then $P(A \mid B) = P(A)$ and $P(B \mid A) = P(B)$.

9. ***Two-way frequency tables*** summarize data in two categories. These tables can be used to show if the two events are independent and to approximate conditional probabilities.

Example:

A random survey was taken to gather information about grade level and car ownership status of students at a school. This table shows the results of the survey.

Car Ownership by Grade

	Owns a Car	**Does Not Own a Car**	**Total**
Junior	6	10	16
Senior	12	8	20
Total	18	18	36

Estimate the probability that a randomly selected student will be a junior, given that the student owns a car.

Solution:

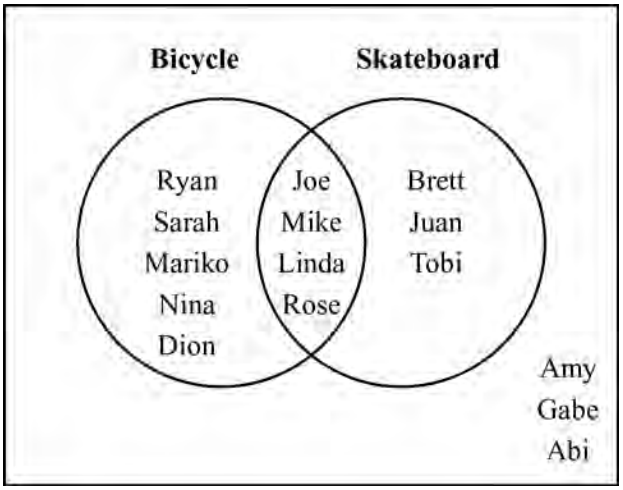
Let $P(J)$ be the probability that the student is a junior. Let $P(C)$ be the probability that the student owns a car.

$$P(J \mid C) = \frac{P(J \text{ and } C)}{P(C)} = \frac{\frac{6}{36}}{\frac{18}{36}} = \frac{6}{18} = \frac{1}{3}$$

The probability that a randomly selected student will be a junior given that the student owns a car is $\frac{1}{3}$.

REVIEW EXAMPLES

1) This Venn diagram shows the names of students in Mr. Leary's class that own bicycles and skateboards.



Let set *A* be the names of students who own bicycles, and let set *B* be the names of students who own skateboards.

a. Find $A \cap B$. What does the set represent?

b. Find $A \cup B$. What does the set represent?

c. Find $(A \cup B)'$. What does the set represent?

Solution:

a. The intersection is the set of elements that are common to both set *A* and set *B*, so $A \cap B$ is {Joe, Mike, Linda, Rose}. This set represents the students who own both a bicycle and a skateboard.

b. The union is the set of elements that are in set *A* or set *B*, or in both set *A* and set *B*. You only need to list the names in the intersection one time, so $A \cup B$ is {Ryan, Sarah, Mariko, Nina, Dion, Joe, Mike, Linda, Rose, Brett, Juan, Tobi}. This set represents the students who own a bicycle, a skateboard, or both.

c. The complement of $A \cup B$ is the set of names that are not in $A \cup B$. So, $(A \cup B)'$ is {Amy, Gabe, Abi}. This set represents the students who own neither a bicycle nor a skateboard.

2) In a certain town, the probability that a person plays sports is 65%. The probability that a person is between the ages of 12 and 18 is 40%. The probability that a person plays sports and is between the ages of 12 and 18 is 25%. Are the events independent? How do you know?

Solution:

Let $P(S)$ be the probability that a person plays sports.
Let $P(A)$ be the probability that a person is between the ages of 12 and 18.
If the two events are independent, then $\cancel{P(S \cup A) = P(S) \cdot P(A)}$ $P(S \cap A) = P(S) \bullet P(A)$.
Because $\cancel{P(S \cup A)}$ $P(S \cap A)$ is given as 25%, find $P(S) \bullet P(A)$ and then compare.

$$P(S) \bullet P(A) = 0.65 \bullet 0.4$$
$$= 0.26$$

Because $0.26 \neq 0.25$, the events are not independent.

3) A random survey was conducted to gather information about age and employment status. This table shows the data that were collected.

Employment Survey Results

	Age (in Years)	
Employment Status	**Less than 18**	**18 or greater**
Has Job	20	587
Does Not Have Job	245	92

a. What is the probability that a randomly selected person surveyed has a job, given that the person is less than 18 years old?

b. What is the probability that a randomly selected person surveyed has a job, given that the person is greater than or equal to 18 years old?

c. Are having a job (A) and being 18 or greater (B) independent events? Explain.

Solution:

a. Find the total number of people surveyed less than 18 years old: 20 + 245 = 265. Divide the number of people who have a job and are less than 18 years old, 20, by the number of people less than 18 years old, 265: $\frac{20}{265} \approx 0.08$. The probability that a person surveyed has a job, given that the person is less than 18 years old is about 0.08.

b. Find the total number of people surveyed greater than or equal to 18 years old: 587 + 92 = 679. Divide the number of people who have a job and are greater than or equal to 18 years old, 587, by the number of people greater than or equal to 18 years old, 679: $\frac{587}{679} \approx 0.86$. The probability that a person surveyed has a job, given that the person is greater than or equal to 18 years old, is about 0.86.

c. The events are independent if $P\left(A \mid B\right) = P(A)$ and $P\left(B \mid A\right) = P(B)$.

From part (b), $P\left(A \mid B\right) \approx 0.86$.

$P(A) = \frac{607}{944} \approx 0.64$

$P\left(A \mid B\right) \neq P(A)$ so the events are not independent.

<u>EOCT Practice Items</u>

1) In a particular state, the first character on a license plate is always a letter. The last character is always a digit from 0 to 9.

If *V* represents the set of all license plates beginning with a vowel, and *O* represents the set of all license plates that end with an odd number, which license plate belongs to the set $V \cap O'$?

A. 

B. 

C. 

D. 

[Key: A]

2) Which of the following events are independent given *P*(*A*), *P*(*B*), and *P*(*A* and *B*)?

A. $P(A) = 0.25$; $P(B) = 0.25$; $P(A \text{ and } B) = 0.5$

B. $P(A) = 0.08$; $P(B) = 0.4$; $P(A \text{ and } B) = 0.12$

C. $P(A) = 0.16$; $P(B) = 0.24$; $P(A \text{ and } B) = 0.32$

D. $P(A) = 0.3$; $P(B) = 0.15$; $P(A \text{ and } B) = 0.045$

[Key: D]

3) Assume that the following events are independent:

- **The probability that a high school senior will go to college is 0.72.**
- **The probability that a high school senior will go to college and live on campus is 0.46.**

What is the probability that a high school senior will live on campus, given that the person will go to college?

A. 0.26

B. 0.33

C. 0.57

D. 0.64

[Key: D]

4) A random survey was conducted about gender and hair color. This table records the data.

Hair Color

	Brown	**Blonde**	**Red**
Male	548	876	82
Female	612	716	66

What is the probability that a randomly selected person has blonde hair, given that the person selected is male?

A. 0.51

B. 0.55

C. 0.58

D. 0.63

[Key: C]

USE THE RULES OF PROBABILITY TO COMPUTE PROBABILITIES OF COMPOUND EVENTS IN A UNIFORM PROBABILITY MODEL

KEY IDEAS

1. Two events are ***mutually exclusive*** if the events cannot occur at the same time.

2. When two events *A* and *B* are mutually exclusive, the probability that event *A* or event *B* will occur is the sum of the probabilities of each event: *P* (*A* or *B)* = *P* (*A*) + *P* (*B*).

3. When two events *A* and *B*, are not mutually exclusive, the probability that event *A* or *B* will occur is the sum of the probability of each event minus the overlap of the two events. That is, *P* (*A* or *B)* = *P* (*A*) + *P* (*B*) – *P*(*A* and *B*).

4. You can find the conditional probability, $P(A \mid B)$, by finding the fraction of *B*'s outcomes that also belong to *A*.

Example:

Event *A* is choosing a heart card from a standard deck of cards.
Event *B* is choosing a face card from a standard deck of cards.
$P(A \mid B)$ is the probability that a card is a heart, given that the card is a face card. You can look at *B*'s outcomes and determine what fraction belongs to *A*; there are 12 face cards, 3 of which are also hearts:
$P(A \mid B) = \frac{3}{12} = \frac{1}{4}$.

	Heart	Not a heart	Total
Face card	3	9	12
Not a face card	10	30	40
Total	13	39	52

REVIEW EXAMPLES

1) In Mr. Mabry's class, there are 12 boys and 16 girls. On Monday, 4 boys and 5 girls were wearing white shirts.

   a. If a student is chosen at random from Mr. Mabry's class, what is the probability of choosing a boy or a student wearing a white shirt?

   b. If a student is chosen at random from Mr. Mabry's class, what is the probability of choosing a girl or a student not wearing a white shirt?

Solution:

a. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, where A is the set of boys and B is the set of students wearing a white shirt.

   A and B is the set of boys wearing a white shirt. There are 12 + 16 = 28 students in Mr. Mabry's class.

   So, $P(A) = \frac{12}{28}$, $P(B) = \frac{4+5}{28} = \frac{9}{28}$, and $P(A \text{ and } B) = \frac{4}{28}$.

   $P(\text{a boy or a student wearing a white shirt}) = \frac{12}{28} + \frac{9}{28} - \frac{4}{28} = \frac{17}{28}$.

b. Apply the Addition Rule, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$, where A is the set of girls and B is the set of students not wearing a white shirt.

   A and B is the set of girls not wearing a white shirt. There are 12 + 16 = 28 students in Mr. Mabry's class.

   So, $P(A) = \frac{16}{28}$, $P(B) = \frac{8+11}{28} = \frac{19}{28}$, and $P(A \text{ and } B) = \frac{11}{28}$.

   $P(\text{a girl or a student not wearing a white shirt}) = \frac{16}{28} + \frac{19}{28} - \frac{11}{28} = \frac{24}{28} = \frac{6}{7}$.

2) Terry has a number cube with sides labeled 1 through 6. He rolls the number cube twice.

   a. What is the probability that the sum of the two rolls is a prime number, given that at least one of the rolls is a 3?

   b. What is the probability that the sum of the two rolls is a prime number or at least one of the rolls is a 3?

Solution:

a. This is an example of a mutually exclusive event. Make a list of the combinations where at least one of the rolls is a 3. There are 11 such pairs.

1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
3, 1	3, 2	3, 4	3, 5	3, 6	

Then identify the pairs that have a prime sum.

2, 3	3, 2	3, 4	4, 3

Of the 11 pairs of outcomes, there are 4 pairs whose sum is prime. Therefore, the probability that the sum is prime of those that show a 3 on at least one roll is $\frac{4}{11}$.

b. This is an example of events that are NOT mutually exclusive. There are 36 possible outcomes when rolling a number cube twice.

List the combinations where at least one of the rolls is a 3.

1, 3	2, 3	3, 3	4, 3	5, 3	6, 3
3, 1	3, 2	3, 4	3, 5	3, 6	

P(at least one roll is a 3) = $\frac{11}{36}$

List the combinations that have a prime sum.

1, 1	1, 2	1, 4	1, 6
2, 1	2, 3	2, 5	
3, 2	3, 4		
4, 1	4, 3		
5, 2	5, 6		
6, 1	6, 5		

P(prime sum) = $\frac{15}{36}$

Identify the combinations that are in both lists.

2, 3	3, 2	3, 4	4, 3

The combinations in both lists represent the intersection. The probability of the intersection is the number of outcomes in the intersection divided by the total possible outcomes.

P(at least one roll is a 3 **and** a prime sum) = $\frac{4}{36}$.

If two events share outcomes, then outcomes in the intersection are counted twice when the probabilities of the events are added. So you must subtract the probability of the intersection from the sum of the probabilities.

P(at least one roll is a 3 **or** a prime sum) = $\frac{11}{36} + \frac{15}{36} - \frac{4}{36} = \frac{22}{36} = \frac{11}{18}$.

<u>EOCT Practice Items</u>

1) Mrs. Klein surveyed 240 men and 285 women about their vehicles. Of those surveyed, 155 men and 70 women said they own a red vehicle. If a person is chosen at random from those surveyed, what is the probability of choosing a woman or a person that does NOT own a red vehicle?

A. $\frac{14}{57}$

B. $\frac{71}{105}$

C. $\frac{74}{105}$

D. $\frac{88}{105}$

[Key: C]

2) Bianca spins two spinners that have four equal sections numbered 1 through 4. If she spins a 4 on at least one spin, what is the probability that the sum of her two spins is an odd number?

A. $\frac{1}{4}$

B. $\frac{7}{16}$

C. $\frac{4}{7}$

D. $\frac{11}{16}$

[Key: C]

3) Each letter of the alphabet is written on a card using a red ink pen and placed in a container. Each letter of the alphabet is also written on a card using a black ink pen and placed in the same container. A single card is drawn at random from the container. What is the probability that the card has a letter written in black ink, the letter A, or the letter Z?

A. $\frac{1}{2}$

B. $\frac{7}{13}$

C. $\frac{15}{26}$

D. $\frac{8}{13}$

[Key: B]

Appendix A
EOCT Sample Overall Study Plan Sheet

Here is a sample of what an OVERALL study plan might look like. You can use the Blank Overall Study Plan Sheet in Appendix B or create your own.

Materials/Resources I May Need When I Study:
(You can look back at page 6 for ideas.)

1. *This study guide*
2. *Pens/pencils*
3. *Highlighter*
4. *Notebook*
5. *Dictionary*
6. *Calculator*
7. *Mathematics textbook*

Possible Study Locations:

- First choice: *The library*
- Second choice: *My room*
- Third choice: *My mom's office*

Overall Study Goals:

1. *Read and work through the entire study guide.*
2. *Answer the sample questions and study the answers.*
3. *Do additional reading in a mathematics textbook.*

Number of Weeks I Will Study: *6 weeks*

Number of Days a Week I Will Study: *5 days a week*

Best Study Times for Me:

- Weekdays: *7:00 p.m. – 9:00 p.m.*
- Saturday: *9:00 a.m. – 11:00 a.m.*
- Sunday: *2:00 p.m. – 4:00 p.m.*

Appendix B
Blank Overall Study Plan Sheet

Materials/Resources I May Need When I Study:
(You can look back at page 6 for ideas.)

1. ____________________________________
2. ____________________________________
3. ____________________________________
4. ____________________________________
5. ____________________________________
6. ____________________________________

Possible Study Locations:

- First choice: ____________________________________
- Second choice ____________________________________
- Third choice ____________________________________

Overall Study Goals:

1. _________________________________
2. _________________________________
3. _________________________________
4. _________________________________
5. _________________________________

Number of Weeks I Will Study: __________________________

Number of Days a Week I Will Study: _____________________

Best Study Times for Me:

- Weekdays: ____
- Saturday: ____
- Sunday: ____

Appendix C
EOCT Sample Daily Study Plan Sheet

Here is a sample of what a DAILY study plan might look like. You can use the Blank Daily Study Plan Sheet in Appendix D or create your own.

Materials I May Need Today:

1. *Study guide*
2. *Pens/pencils*
3. *Notebook*

Today's Study Location: *The desk in my room*

Study Time Today: *From 7:00 p.m. to 8:00 p.m. with a short break at 7:30 p.m.*
(Be sure to consider how long you can actively study in one sitting. Can you sit for 20 minutes? 30 minutes? An hour? If you say you will study for three hours, but get restless after 40 minutes, anything beyond 40 minutes may not be productive—you will most likely fidget and daydream your time away. "Doing time" at your desk doesn't count as real studying.)

If I Start to Get Tired or Lose Focus Today, I Will: *Do some sit-ups*

Today's Study Goals and Accomplishments: (Be specific. Include things like number of pages, units, or standards. The more specific you are, the better able you will be to tell if you reached your goals. Keep it REALISTIC. You will retain more if you study small "chunks" or blocks of material at a time.)

Study Task	*Completed*	*Needs More Work*	*Needs More Information*
1. Review what I learned last time	X		
2. Study the first main topic in Unit 1	X		
3. Study the second main topic in Unit 1		X	

What I Learned Today:

1. *Reviewed basic functions*
2. *The importance of checking that the answer "makes sense" by estimating first*
3. *How to use math symbols*

Today's Reward for Meeting My Study Goals: *Eating some popcorn*

Appendix D
Blank Daily Study Plan Sheet

Materials I May Need Today:

1. ________________________________
2. ________________________________
3. ________________________________
4. ________________________________
5. ________________________________

Today's Study Location: ______________________

Study Time Today: ______________________

(Be sure to consider how long you can actively study in one sitting. Can you sit for 20 minutes? 30 minutes? An hour? If you say you will study for three hours, but get restless after 40 minutes, anything beyond 40 minutes may not be productive—you will most likely fidget and daydream your time away. "Doing time" at your desk doesn't count as real studying.)

If I Start To Get Tired or Lose Focus Today, I Will: ________________________________

Today's Study Goals and Accomplishments: (Be specific. Include things like number of pages, sections, or standards. The more specific you are, the better able you will be to tell if you reached your goals. Keep it REALISTIC. You will retain more if you study small "chunks" or blocks of material at a time.)

Study Task	*Completed*	*Needs More Work*	*Needs More Information*
1.			
2.			
3.			
4.			
5.			

What I Learned Today:

1. ____
2. ____
3. ____

Today's Reward for Meeting My Study Goals: ____